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Efficient agglomeration of spatial clubs ☆

Oded Hochman

Department of Economics, Ben-Gurion University of the Negev, Beer Sheva 84105, Israel

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ABSTRACT

We investigate here the agglomeration of spatial clubs in an efficient allocation of a club economy. The literature on agglomeration has focused largely on a primary agglomeration caused by direct attraction forces. We concentrate mainly on secondary and tertiary agglomerations caused by a primary agglomeration. Initially, scale economies in the provision of club goods (CGs) lead each CG to agglomerate in facilities of its club. This primary agglomeration causes a secondary concentration of population around these facilities, which in turn brings about a tertiary agglomeration of facilities of different clubs into centers in the midst of population concentration. The agglomeration of facilities occurs only if a secondary concentration of population takes place. We analyze in detail two specific patterns of agglomeration. One is the central location pattern in which the facilities of all clubs agglomerate perfectly in the middle of the complex. The second is a triple-centered complex in which the center in the middle of the complex consists of perfectly agglomerated facilities of different clubs, each with a single facility per complex. The remaining two centers also consist of facilities of different clubs, but clubs in these centers each have two facilities per complex, one in each center. Each of these two centers is located between a boundary and the middle of the complex closer to the middle of the complex than to the boundary. The facilities in these two centers form condensed clusters of facilities that may contain residential land in between the facilities. We then show that these agglomeration patterns also characterize agglomerations in general. The literature maintains that an efficiently behaving municipality increases its tax-base. This implies that it is in the municipality's interest to achieve efficiency. The best way for a local government to achieve this desired efficiency is by partially intervening in market operations in order to internalize local externalities. Such an intervention should be limited to providing the city's infrastructure, to taxing only residential land rents and clubs' profits, to subsidizing the basic industry of the city, and to partially regulating land uses. Consequently, if the local governments of all complexes behave according to the above, the decentralization of the efficient allocation of the club economy would be attained.

1. Introduction

The purpose of this paper is threefold: the first is to introduce an optimization model of an economy with spatial clubs, the second is to identify those forces in the economy that lead to the agglomeration of facilities of various clubs into multi-club centers, and the last is to characterize the solution in general and these centers in particular. To facilitate the exposition, we first introduce some terminology related to the theory of spatial clubs. A spatial club consists of facilities spread throughout the economy, each of which contains a concentration of the good provided by the club. A club-good (CG) is a good or service provided by each of the club facilities to their patrons. The provision of a CG by its club's facility is subject to scale economies. The patrons of a facility are a group of households who jointly consume the CG provided by the facility and are distinct from patrons of other facilities of the same club. In order to consume a particular CG, a household has to commute to one of the facilities of the spatial club that provides this good. The market area of a facility is the area of residency of the facility's patrons.

Many local public goods are CGs as are many private consumption goods and services whose provision is subject to scale economies and therefore are provided collectively by spatial clubs. Most clubs belong to the type of clubs to which people commute, which include: Real-life clubs such as country clubs, parks, museums, churches, etc. In addition, other institutions, not necessarily known as clubs, satisfy our specifications, including schools (e.g. see Jepsen and Montgomery (2009) who show the importance of distance to a community college), police stations, theater and movie halls, restaurants, government offices, courthouses, shops and stores, to name just a few. Notable among these various clubs is the 'production club', in which the population is employed. The real-life facilities of the production club

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E-mail address: oded@bgu.ac.il

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include industrial areas and employment centers. Clubs that deliver goods or services from facilities to household homes are a second type of real-life spatial club that fulfill the model's specifications. Delivery costs play the same role in this type of club that transportation costs played in clubs of the type to which people commute. Examples of this type of club would be electricity utilities whose facilities are transformer stations and water utilities whose facilities are pumping stations. Another example of a club delivering home its CG is fire-protection services where fire fighting stations are the facilities.

Three main causes of residential and non-residential agglomerations are typically cited in the literature. One results from reciprocal informational exchange, the second from increasing returns to scale and the last from spatial competition (for theories on agglomeration see Fujita and Thisse (1996, 2002) and for a review of the substantial economic literature on this topic see Rosenthal and Strange (2004)). Most of these explanations of agglomerations are based on direct attraction forces such as the mutual attraction of units of an industry because their activity is enhanced when located close to each other. In this paper, the primary agglomeration of CGs into facilities is a result of a direct attraction between units of a CG whose provision is subject to scale economies. Each CG agglomerates into its own facilities in order to provide the CG to households throughout the economy. We focus here, however, mainly on the secondary concentrations of population around facilities and on the tertiary agglomerations of facilities of different clubs in centers in the midst of population concentrations.

The primary agglomeration of a CG in facilities attracts households to locate close to a facility in order to save commuting costs. The desire to save commuting costs is offset by congestion costs due to the limited supply of land in the proximity of the facility. The indirect attraction and the subsequent congestion cause a secondary concentration of population around facilities, where the density of population decreases with its distance from the facility. In turn, the concentration of population around a facility causes facilities of different clubs to locate in the same vicinity in order to increase accessibility even further, thus creating tertiary agglomerations of facilities into centers in the midst of densely populated areas. All three stages of agglomeration, namely the primary agglomeration of CGs, the secondary concentration of population and the tertiary agglomeration of facilities into centers, occur simultaneously and the stages indicate the order of causality rather than the timing. Indeed, we show that tertiary agglomeration does not occur without a secondary concentration of population and that secondary concentration of population does not occur without the primary agglomeration of CGs into facilities.[1]

Early urban economics models dealt mainly with secondary concentration of households in a residential ring surrounding a predetermined central business district (CBD), where all employment takes place. The concentration of industry in the CBD was exogenously assumed, the rationale being that the industry must be located in proximity to a seaport, train depot or other shipping facility through which the city's basic products can be exported to the rest of the world (see Muth (1969)). Mills (1967) and Mills and Hamilton (1989) argued that the agglomeration of industry in a CBD is the result of the industry being subject to scale economies but they still assumed exogenous agglomeration. Instead of focusing on an endogenous CBD, Mills and his contemporaries concentrated on the residential ring. Henderson (1974) was the first to introduce a model in which an industry agglomerates endogenously into a CBD, however, he still imposed a single employment location surrounded by a residential ring on the model. In subsequent studies, Ogawa and Fujita (1980), Fujita and Ogawa (1982), and Fujita (1989) constructed simulation models of the agglomeration of an industry based on direct attraction effects. These simulations resulted in a variety of primary agglomerations. However, no secondary concentration of population and hence no tertiary agglomerations were possible, since a uniform density of population was assumed everywhere. The new economic geography model by Fujita et al. (1999) presented a formal study on the evolution of the central place systems in which concentration of workers and agglomeration of multiple types of industries play a central role. Recently, Lucas and Rosi-Hansberg (2002) incorporated both direct and indirect agglomeration engines into a single simulation model of an agglomerating industry and population/workers. But, contrary to our model in which facilities of different clubs agglomerate into centers, in their model only one type of facility exists and therefore no tertiary agglomeration can occur.

The above models do not address the endogenous tertiary agglomeration of different primary agglomerations into centers in the midst of population concentrations as described in this paper because they either have only a single industry which can agglomerate or they do not have secondary agglomerations. Some studies (e.g., Fujita and Thisse (1986), Thisse and Wildasin (1992), Papageorgiou and Pines (1999) and papers surveyed by Berliant and ten Raa (1994)) do investigate agglomeration of different facilities but they impose a uniform distribution of population on the model. We show here that without a secondary concentration of population an agglomeration of facilities is ineffective.[2] The agglomerations of facilities in the above studies are due to either the 'edge-of-economy effect,' to indivisibility and/or to random technological effects. Therefore, to avoid confounding our own results, we assume herein an economy without edges, i.e., our economy's territory is ring-shaped and fully occupied. In addition, we investigate only cases of full divisibility.

On this ring-shaped area of homogeneous land, we construct a model of an economy with spatial clubs using the conceptual framework of Hochman et al. (1995).[3] In this economy there are many types of essential collective goods that require a wide variety of spatial clubs that a household must visit in order to consume the goods. The concentration of each CG into a separate facility results from scale economies in the provision of the good. Without such scale economies, each household would consume the CG privately in its own premises to avoid commuting costs. Since the direct attraction forces between units of a CG caused by scale economies are assumed to be internal to the facility, they are reflected only in

[1] In real-life situations there are additional reasons that are not captured by this model for various types of spatial clubs to operate close to each other. These reasons include: (1) multi-purpose trips intended to save commuting costs, i.e., one trip to two or more different club facilities; (2) joint services to users of different facilities that are subject to scale economies, for instance: joint infrastructures such as parking lots, rest areas, rest rooms, etc. (3) The advantages in (1) and (2) require that patrons should spend a relatively long time at each trip, which, in turn, implies that different R&R facilities should be included in the site to encourage extended stays. The above three points may explain the existence of large suburban malls that our model does not explain. (4) Clubs operating during different hours may use the same infrastructure at different times. For instance, employment centers operating in the central business district (CBD) during week days at the daytime while entertainment clubs like nightclubs, restaurants, bars etc., operate weekends and during the night. Therefore, both may use the same roads and parking space at different times of the day or the week. This may explain why an 'Old City' is located in the midst of a CBD in some cities, or why a wedding hall exists in the midst of industrial parks. These effects cannot be investigated within the framework of our model.

[2] An agglomeration is 'ineffective' when in addition to the original allocation with agglomeration there is an equivalent allocation which solves the model that has no agglomeration (see Section 5.1).

[3] While Hochman et al. (1995) focused on the finance of services rendered by the facilities, they disregarded spatial aspects and questions of agglomeration of facilities, on which the present paper focuses.

the size of the facilities and not in their number. Thus, at any given site no more than one facility per club exists. Then we demonstrate that the population density is never uniform in a first-best allocation and that there are always areas in the economy in which population and facilities agglomerate.

Our model's results specify that in an optimal allocation the economy's territory is partitioned into identical *complexes*, where a complex is the smallest autonomous area in the economy, i.e., the smallest area in which all residents, and they alone, consume all the types of CGs in facilities located inside the complex. Thus, nobody commutes in or out of a complex, which, in a sense, renders the complex the ideal municipality. Furthermore, we characterize an allocation by characterizing its representative complex.

A *complex configuration* is a vector of integers whose greatest common divisor (GCD) is one.[4] Each entry in the vector specifies the number of facilities of a club in the complex. Thus, the first entry in the vector is the number of facilities of club one in a complex, the second entry is the number of facilities of club two and so forth. The entries of a complex configuration are integer variables that have to be solved endogenously. However, being integer variables makes them hard to solve by using regular analysis. We therefore first solve the model for a given constant complex configuration to which we refer as a local optimum. In a global optimum the complex configuration is also chosen optimally. We show that in the optimum there is always a solution with a symmetric complex.

Next, we characterize the spatial pattern of two local optimum solutions of two specific complex configurations.[5] In the first configuration, each club has a single facility per complex. With this configuration, the model results in monocentric complexes (cities) in which facilities of all clubs agglomerate perfectly in the center of the complex and share the whole complex as a common market area. [6] The population density and the housing price function in each of the complexes of this configuration increase with proximity to the complex's center, where both functions reach their peak. In addition, we provide specifications of a functions' domain in which this solution is the unique global (over all possible configurations) optimum.

The second configuration that we investigate has two groups of clubs. Each club in the first group has a single facility per complex and each club in the second group has two facilities per complex. In the optimal allocation, all the facilities of clubs of the first group agglomerate perfectly in the middle of each complex with their market area consisting of the whole complex. The facilities of clubs of the second group are divided into two clusters, each of which contains one facility of each club of the second group. The complex area is divided in the middle into two equal market areas, one for each cluster of facilities of the second group of clubs. One cluster is located in the second quarter of the complex's area and the other in the third quarter. Thus, the clusters of the second group are closer to the middle of the complex than to its boundaries. In other words, these clusters gravitate towards the center of the complex. The facilities in a cluster are close to each other but residential areas may exist between the facilities in the cluster, depending on whether or not the transportation cost functions of the different clubs with two facilities per complex (DF hereafter) are proportional to each other. Facilities with proportional transportation costs share the same facility location. Thus, while clubs of the second group do not necessarily agglomerate perfectly, they are drawn to each other and form clusters, which as a whole are drawn towards the center of the complex. The complex is symmetric around its middle with a higher density of population between the clusters of the DF clubs and the center of the complex than between the clusters and the boundaries. We then generalize part of the above results for complexes in general and show that in our model these two types of agglomerations epitomize agglomerations of facilities in general.

Contrary to non-spatial clubs (e.g., Berglas (1976), Scotchmer and Wooders (1987); see also the survey by Scotchmer (2002) of spatial and non-spatial clubs and Waldfogel (2008)), our optimal solution cannot be attained by a laissez faire allocation. In a laissez faire situation club owners are free to operate without restrictions, so they engage in spatial monopolistic competition, which in general does not yield an optimal allocation (e.g., Beckmann, 1999). In Hochman et al. (1995) it is claimed that to attain the efficient allocation, the local government has to provide by itself the club goods to the general population. Here we argue that there is a decentralization method coupled with regulations with which the optimum may be approached with only a limited government intervention. In such a decentralization, households pay facility operators for the use of their CG and the local government taxes away the operators' profits and determines the location of the facilities.

Five sections follow this introduction. Section 2 describes the setup of the model. The necessary conditions for Pareto optimum are described in Section 3 and the decentralization of the optimal allocation is depicted in Section 4. Section 5 contains our main results. In Section 5.1, we present general characteristics of the solution. In Section 5.2, we describe a perfect agglomeration and in Section 5.3, an imperfect agglomeration. We generalize the results in Section 5.4 and in Section 5.5, we elaborate on how to obtain global optima with global configurations. In Section 6, we conclude with a short summary and a few pointers for future research.

2. The model setup

The country's geography is designated by a ring of unit width, with a circle running through the middle of the ring to serve as the ring's x-axis (see Fig. 1). We assume the axis-circle's circumference is $\mathscr{L}$. Note that the total area of the ring in this case is also $\mathscr{L}$. An arbitrary point on the ring's axis is referred to as the origin. The location of any point on the axis of the ring is uniquely defined by its distance x from the origin in a clockwise direction (henceforth also the positive or the right direction). All points on the line segment perpendic-

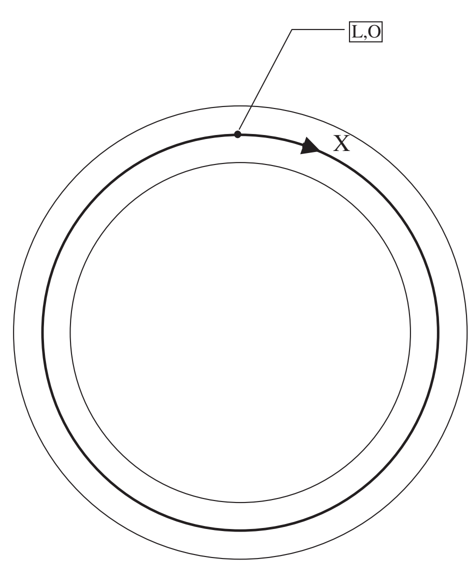

Fig. 1. A ring-shaped economy.

[4] The GCD of a vector of integers is the largest integer that divides all the entries of the vector and leaving each of them still an integer. When the GCD of a configuration is one the configuration consists of disjoint integers and is termed basic.

[5] These complex configurations are: (1,...,1) and (1,...,1, 2,...,2).

[6] By perfect agglomeration we mean that facilities are adjacent to each other with no residential area between them.

ular to the axis are designated as the same location because travel between these points involves no costs. The country accommodates $\mathcal{N}$ *households* (each time we introduce a concept it is italicized) which are identical to each other in all respects. We assume that these households are free to choose their residential location in the economy. Hence, all households must have the same utility level everywhere; otherwise they will migrate to the location with the higher utility. Each individual household derives utility from the consumption of a *composite good*, Z, and from *housing*, H, both of which the household consumes at its location of residency.

The household also derives utility from I types of *collective goods* (CGs hereafter), where G_i, is the quantity of the ith CG the household consumes, $i = 1,\ldots,I$, according to a well-behaved utility function, $u(Z,H,G_1,\ldots,G_I)$. All goods are *essential*, and each CG is consumed at a special facility to which the household has to travel. Each individual is endowed with Y units of the composite good which can be used for private consumption and for the production of housing, CGs and transportation.

The economy contains I different clubs, one for each type of CG. A *club* of type i supplies units of the ith CG through its $\tilde{m}_i$ *facilities* which are located throughout the economy. Each facility is identified by i, j, where $j \in (1,\ldots,\tilde{m}_i)$ is the index of the specific facility of club i, and $i \in (1,\ldots,I)$ refers to the club type. Facility i, j, whose location is designated by $x_{i,2j}$, provides G_{ij} units of the ith CG to N_{ij} *patrons*, i.e., to individual households consuming the ith CG in facility ij and residing within its *market area*, where a market area of a facility is a segment of the x-axis in which all and only the facility's patrons live.[7] We also make the simplifying assumption that a facility does not occupy land and since, in practice, club facilities occupy only a small fraction of the total land available compared to residential land, the distortion caused by this assumption is negligible when considering the simplification involved. We represent facility ij's market area by the interval $[x_{i,2j-1},x_{i,2j+1}]$. The union of the market areas of the $\tilde{m}_i$ facilities supplying the ith CG coincides with the *residential area* $[0,\mathsf{L}]$ where L, the boundary of the residential area, fulfills the condition that $\mathsf{L} \leqslant \mathcal{L}$.[8] Accordingly, the spatial characteristics of each facility ij are fully specified by the following triplet of nodes (see Fig. 2):

$x_{i,2j-1}$ = the left boundary of the ijth facility's market area and the right boundary of the $i(j-1)$th facility's market area,
$x_{i,2j}$ = the location of the ijth facility, and
$x_{i,2j+1}$ = the right boundary of the ijth facility's market area and the left boundary of the $i(j+1)$th facility's market area.

Since all goods are essential, each resident must consume all types of club goods. Hence, the extreme boundaries must fulfill, $x_{i,2\tilde{m}_i+1} = \mathsf{L}$, and $x_{i,1} = 0$, $\forall i$.[9] We define the clubs' configuration as the vector of integers $(\tilde{m}_1,\ldots,\tilde{m}_I)$, where $\tilde{m}_i$ is the number of facilities of type i in the economy. Thus, the clubs' configuration is a vector of I integer variables. To facilitate the analysis, we sort all the possible configurations into classes, where each class is represented by a vector $(m_1,\ldots,m_I)$ ((m_i) for brevity) of I disjoint integers whose greatest common divisor (GCD henceforth) is one, i.e., for every

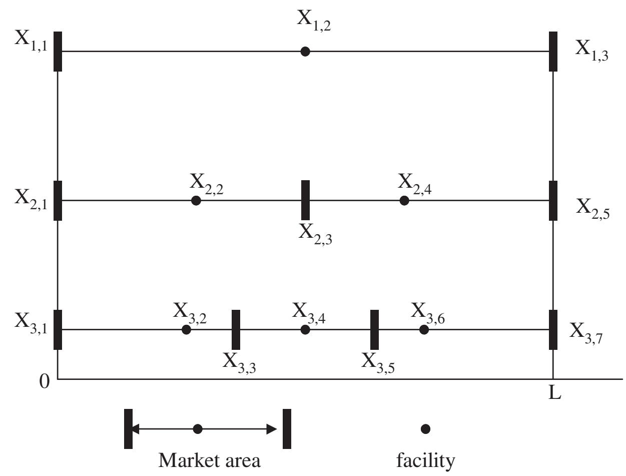

Fig. 2. Facility locations and market areas in a complex with a basic configuration (1,2,3).

$\lambda \geqslant 2$, at least one of the quotients m_i/λ, $i = 1,\ldots,I$, is not an integer. We term the configuration with $GCD = 1$ a *basic configuration.* From here on, we designate a clubs' configuration $(\tilde{m}_i)$ by $k(m_i)$, where (m_i) is the basic configuration designating the class, and the GCD k is an additional integer-variable to be solved.

Figure 2 depicts the layout of a complex with a basic configuration of (1,2,3). For expositional purposes, we mark the nodes of each club on a different horizontal axis. Actually, they are all jointly located on the x-axis.

Next, we consider a model of an economy with population $\mathcal{N}$, available land $\mathcal{L}$ and a clubs' configuration $k(m_i)$. A complex in this economy is the optimal solution of a model whose population size is $\frac{\mathcal{N}}{k}$ and its available land is $\frac{\mathsf{L}}{k}$, where $\mathsf{L} \leqslant \mathcal{L}$ is the occupied land in the original economy. The clubs' configuration of the complex model is the basic (m_i) and it has the same functions (costs, utility) as the original model. In the solution of the complex, the GCD, k, is 1, the configuration is the basic configuration (m_i) and all its land, $\frac{\mathsf{L}}{k} \overset{\text{def}}{=} L$, is occupied by $\frac{\mathcal{N}}{k} \overset{\text{def}}{=} N$ households. The optimal solution of the model with the configuration $k(m_i)$ can now be described as k consecutive replications of the complex with the basic configuration (m_i). Each two consecutive complexes are adjacent and have a joint boundary. $\mathcal{L} - \mathsf{L} \geqslant 0$ is the vacant land at the edges. The GCD k, is now an integer-variable measuring the number of complexes in the economy. Thus, by determining k and characterizing the complex, we characterize the solution of the general model.[10] In the rest of the paper, we use the terms basic configuration and complex configuration interchangeably. Since L is the length (also the area) of the complex and the population of the complex is N, $kL = \mathsf{L}$ and $kN = \mathcal{N}$. Accordingly, L is also the coordinate of the right boundary of the first complex (whose left boundary is the origin, 0) and the left boundary of the second complex, if there is more than one complex in the economy, and so on for all k complexes. Since all complexes are identical, it is sufficient to solve only for one (the first) complex. In addition, because all goods are essential, the boundaries of a complex must coincide with the boundaries of each of the facilities of the I clubs that are farthest from the complex center. Hence,

$$x_{i,1} = 0; \quad L - x_{i,2m_i+1} = 0, \quad \forall i \in \{1,\ldots,I\} \text{ and } kL = \mathsf{L} \leqslant \mathcal{L}. \tag{1}$$

[7] By this we assume that a market area of a facility is a connected segment. In what follows we prove that, indeed, the market area of a facility of a club is a connected segment, provided that $t_i(x)$, the club's commuting cost function, is linear in x (see Lemma 3). In the case of non-linear transportation costs, connected market areas remain an assumption.

[8] By this, we make the assumption that the occupied area as well as the unoccupied area are connected and that the later is concentrated at the end of the economy between L and the origin, 0.

[9] In this model, the focus is on the case in which all available land is occupied, i.e., $\mathsf{L} = \mathcal{L}$, which implies that $0 \equiv x_{i,1} = x_{i,2\tilde{m}_i+1} = \mathsf{L} = \mathcal{L}$, $\forall i$. Therefore, calculations with the location variable x are modulo $\mathcal{L}$ (i.e. $\mathcal{L} + x = x$.). For example, $\forall i$, $\tilde{m}_i$ and an arbitrary y, $0 < y < \mathcal{L}, x_{i,1} + y = x_{i,2\tilde{m}_i+1} + y = \mathcal{L} + y = y$.

[10] In Hochman et al. (1995) a complex is defined as the smallest autonomous area in the economy, i.e., the smallest area in which its residents, and only them, consume the CGs in it From the discussion so far it is clear that our complex satisfies this definition.

Eq. (1) implies that by assumption the origin is a boundary of all clubs. Similarly, the relation between the complex and the overall population must be

$$\mathcal{N}/k - N = 0. \tag{2}$$

In order to use a CG, the household incurs travel costs of a home-facility trip which is given by $t_i(|x - x_{i,2j_i(x)}|)$, where the argument of the function is the absolute value of the home-facility distance and $j^i(x)$ is the index j of the facility of club i whose residents at x use. The transportation cost function fulfills $t_i'(y) > 0, t_i''(y) \leqslant 0$, for all $y \geqslant 0$.[11]

The *provision cost function*, $c^i(G_{ij}, N_{ij})$ (for brevity, hereafter $c^i(j)$) is the cost to facility i, j for providing its CG, G_{ij}, to N_{ij} households. The function $c^i(j)$ fulfills,

$$c_1^i(j) = \frac{\partial c^i(j)}{\partial G_{ij}} > 0, \quad c_2^i(j) = \frac{\partial c^i(j)}{\partial N_{ij}} \geqslant 0, \quad c_{11}^i(j) = \frac{\partial^2 c^i(j)}{\partial G_{ij}^2} > 0,$$

$$\partial\left(\frac{c^i(j)}{N_{ij}}\right)/\partial N_{ij}\begin{cases} < 0 & \text{if } N_{ij} < \overline{N}_{ij}(G_{ij}), \\ \geqslant 0 & \text{if } N_{ij} \geqslant \overline{N}_{ij}(G_{ij}), \end{cases} \quad \partial^2\left(\frac{c^i(j)}{N_{ij}}\right)\Big/\partial(N_{ij})^2 > 0$$

$$\text{where } 0 < \overline{N}_{ij}(G_{ij}) \leqslant \infty, \quad \text{and } G_{ij} \geqslant 0. \tag{3}$$

Thus, $\frac{c^i(j)}{N_{ij}}$ is either a *U*- or *L*-shaped function of N_{ij}.[12] We designate by $\theta = \frac{\partial c}{\partial N}\frac{N}{c}$ the congestability level of a CG, $0 \leqslant \theta < 1$, when $\theta = 0$ the CG is a pure LPG and when $\theta = 1$ the CG is a private good. The scale economies reflected in the second line of (3) are responsible for the concentration of club goods in facilities. Without these scale economies, a CG would be provided to a household, like *z*, at home and not in facilities where there is joint consumption of households. Each facility i, j is identified by its CG, G_{ij}, facility location, $x_{i,2j}$, market area, $(x_{i,2j-1}, x_{i,2j+1})$ and the population within its market area, N_{ij}. A club that requires special attention is the 'production club', which we designate by the index $i = 1$.[13] Patrons (workers) N_{1j}, of facility $1j$ of a production club work in the club's facility location $x_{1,2j}$, reside in the facility's market area and, together with an input of G_{1j} units of composite good, produce a net positive output $(-c^1(G_{1j}, N_{1j}) > 0)$ of the composite good. Thus, $[G_{1j} - c^1(G_{1j}, N_{1j})]$ is the gross output of the jth facility of club 1 and as such, is its production function. The general characteristics of a club's cost functions specified in (3) need some modification and interpretation in the case of a production club. Thus, instead of (3) we assume,

$$\begin{aligned} & c_1^1(j)\begin{cases} < 0, & \text{if } G_{1j} < \overline{G}_1(N_{1j}) \\ \geqslant 0, & \text{if } G_{1j} \geqslant \overline{G}_1(N_{1j}) \end{cases} \quad \text{where} \quad \frac{\partial \overline{G}_{1j}(N_{1j})}{\partial N_{1j}} > 0 \\ & c^1(G_{1j}, 0) = 0, \quad c_2^1(j) < 0, \quad \forall G_{1j}, N_{1j}; \\ & c_{11}^1(j) > 0, \qquad c_{22}^1(j) \geqslant 0, \text{and} \\ & \frac{\partial\left(\frac{\lambda G_{1j} - c^1(\lambda G_{1j}, \lambda N_{1j})}{\lambda N_{ij}}\right)}{\partial \lambda} > 0; \quad \frac{\partial^2\left(\frac{\lambda G_{1j} - c^1(\lambda j)}{\lambda N_{ij}}\right)}{\partial \lambda^2} < 0, \text{for} \quad \lambda \geqslant 1, \end{aligned} \tag{4}$$

Accordingly, for $N_{1j} > 0$, the function $c^1(G_{1j}, N_{1,j})$ obtains negative values and is *U*- shaped as a function of G_{1j}, while the average output, $\left(\frac{G_{1j} - c^1(G_{1j}, N_{1j})}{N_{1j}}\right)$, is increasing when both inputs increase proportionally. This last property is a reflection of scale economies in production.[14] We also assume in the production club case that the marginal utility of G_{1j} is zero, i.e., $\partial u/\partial G_1 = 0$, which means that G_{1j} is a production factor that does not affect the household's well-being.

We adopt here the assumption accepted in urban economics literature of a non-atomic distribution of population over space. Thus, a household in our model is identified by its residence at *x*. In addition, we confine ourselves to allocations in which all households are identical in the sense that they all have the same utility function, skills, and initial endowment and they all face the same transportation and provision cost structure. In that case, free choice of the location of residency implies an equal utility level for everyone everywhere, namely:

$$U - u\left(Z(x), H(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)}\right) \leqslant 0, \quad \forall x \in [0, kL], \tag{5}$$

where U is the common utility level for all households in the economy and $j^i(x)$ is the index of the facility providing the ith CG to households living at x. We designate by $u_i(x)$ the derivative of $u(Z(x), H(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)})$ with respect to the ith variable of the utility function as specified in (5), e.g., $u_2(x) = \frac{\partial u(Z(x), H(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)})}{\partial H(x)}$.

We now turn to housing construction. Let $H^s(x)$ be the amount of housing constructed per unit land at x. $H^s(x)$ is produced by land and the composite good. The amount of composite good used in the production per unit of land at x is $c_h(H^s(x))$, with $c_h'(H^s) > 0$ and $c_h''(H^s) > 0$. We term $c_h(H^s)$ as the *housing cost function*.The material balance for housing implies

$$n(x)H(x) - H^s(x) \leqslant 0, \tag{6}$$

where $n(x)$ is the *population density function*.

The club membership constraint can be written as:

$$N_{ij} - \int_{x_{i,2j-1}}^{x_{i,2j+1}} n(x)dx \leqslant 0, \quad \forall i, j; \begin{cases} i \in \{1, \ldots, I\} \\ j \in \{1, \ldots, m_i\} \end{cases}. \tag{7}$$

and

$$\mathcal{N} - \sum_{j=1}^{m_i} N_{i,j} = 0 \quad \forall i \in \{1, \ldots, I\}. \tag{8}$$

The *housing price function*, $p_h(x)$, is defined as:

$$p_h(x) \stackrel{\text{def}}{=} u_2(x)/u_1(x), \tag{9}$$

where the composite good Z is the numeraire. From (9) and (5) we substitute out $H(x)$ and $Z(x)$ to obtain the *compensated demand function for housing*, namely

$$H(x) = h[p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)}, U], \tag{10}$$

[11] The assumption $t''(y) \leqslant 0$ is accepted in the urban economics literature. The main justification of the assumption are travel congestion costs. The number of travelers on the road increase when the distance from the facility reduces, therefore, congestion costs closer to the facility are higher than farther away from it. Congestion costs are even higher when facilities are located in centers. A second reason for $t''(y) \leqslant 0$ is the fixed costs at the end and beginning of a trip, i.e., the value of walking, waiting and parking time. The assumption of a non-positive second derivative of the travel cost function ensures that the bid housing price function and with it the housing price function are concave (see Appendix A). Note, however, that aggregate travel costs to a facility are a convex function of the size of a facility's patronage.

[12] Note that $c_2^i(j) = 0$ implies that G_{ij} is a pure public good with an L-shaped average cost function. Then $c^i(G, N) = c^i(G, 1)$ for all values of N and G. When G is a private good distributed equally to each of the N residents, $c^i(G, N) = Nc^i(G, 1)$. Accordingly, as long as $c^i(G, N)$ fulfils $c^i(G, 1) < c^i(G, N) < Nc^i(G, 1)$, G behaves as a congestable local public good, i.e. $0 < \theta < 1$.

[13] For simplicity, we assume that all clubs other than the 'production club' do not employ labor. Accordingly, these non-production clubs consist only of composite good and patrons/customers. The production club produces the composite good by using labor. It should be noted that no more than one employment club can exist in the model because of the assumption that every household must visit *each* club in the economy while workers cannot work in more than one workplace. In addition, we assume away in our model important aspects like pollution externalities that Arnott et al. (2008) has shown to matter and preference externalities that Waldfogel (2008) considers important.

[14] In subsequent sections, results specific to the production club are given in footnotes.

and the *compensated demand function for the composite good*, which is

$$Z(x) = z[p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)}, U], \tag{11}$$

where $p_h(x)$ together with the different CGs and the utility level, U, are arguments in both of the above functions. Let the *aggregate expenditure function* for the (representative) complex be given by $E(N,U)$ where

$$E(N,U) = \int_0^L [n(x)z(\cdot) + c_h(H^s)]dx + \sum_{i=1}^{I}\sum_{j=1}^{m_i} c^i(j) + \sum_{i=1}^{I}\sum_{j=1}^{m_i}\int_{x_{i,2j-1}}^{x_{i,2j+1}} n(x)t_i(|x - x_{i,2j}|)dx. \tag{12}$$

The three terms of the complex's aggregate expenditure function in (12) are the expenditures on consumption and housing production (the first term), the provision cost of all CGs (the second term), and the total transportation costs (the third term). Accordingly, $kE(N,U)$ is the economy's aggregate expenditure function.

Recalling that each individual is endowed with Y units of the composite good, the complex's material balance of the composite good requires that

$$E(N,U) - NY \leqslant 0. \tag{13}$$

In other words, the complex's aggregate expenditure must equal the complex's aggregate supply of the composite good.

The above set of Eqs. (1)–(13) defines the constraints of a feasible spatial resource allocation for the whole economy. Necessary conditions for a Pareto optimal allocation are given in the next section.

3. The optimal solution

The necessary conditions for a Pareto optimal allocation in which all individuals in the economy have the same utility level are obtained by maximizing the common utility level, U, subject to the constraints (1)–(13).[15] The Lagrangian and the formal derivation of the first-order conditions are specified in Appendix 8.1 in the Web Appendixes. When solving the model, we assume for simplicity that the variable k, which is the number of complexes in the economy, is a real variable and not an integer. By making this assumption, we disregard the factual indivisibility of complexes and allow a fraction of an optimal complex in the solution.[16] The conditions in this section are necessary for a single complex. In our economy there are k such complexes. We also assume that the complex configuration, $(m_1, \ldots, m_I)$, is a given vector of I integers. Therefore, the necessary conditions below are for a local optimum. Additional conditions for a global optimum, in which the optimal complex configuration is determined as well, follow in a subsequent section, after the local optimum is discussed.

The necessary conditions below are also sufficient for a (local) maximum. Since the constraints in our model are formulated as inequalities and due to the conditions on the model functions as specified in the previous section, the feasible set is compact. Consequently, a solution of the model that satisfies the necessary conditions exists and cannot be a minimum. However, when non-convexities are involved, there may be corner solutions with multiple local optima. Scale economies are such non-convexities and therefore where scale economies are involved a solution includes spatial corner solutions, i.e., separate facilities where a CG is consumed and to which consumers commute in order to use the CG (the alternative internal solution is that the CG is consumed by each household at home). When the complex configuration's entries are fixed and given, an allocation that satisfies the necessary conditions exists for all the functions that fulfill the requirements specified in the previous section. Note that since housing is an essential good, in every solution there must be occupied land. Even for a fixed configuration we cannot rule out the possibility of several local optima due to the non-convexities and it may happen that more than one of these local optima also has the maximum utility level. In such a case the optimum is not unique.

In Section 5.5, we discuss aspects of global optimum solutions and how the feasible set of functions that constitute the functions domain of the fixed-configuration-model is divided into subsets for which only one basic configuration is in the global optimum. We also discuss the global optimum in which the distribution of CGs is at home as a private good and not in facilities.

The equations in this section are calculated from the necessary conditions derived in Appendix 8.1 in the Web Appendixes. These equations are easier to interpret than the original ones and still constitute a full set of necessary conditions for a Pareto optimal complex, equivalent in every way to the original conditions derived in the Appendix.

3.1. Households and housing

3.1.1. Housing construction

In (9), $P_h(x)$ is defined as the quotient $u_2(x)/u_1(x)$. A necessary condition for the efficient allocation given in (14) below states that *the marginal cost of housing construction*, $c'_h(H^s(x))$, equals $P_h(x)$, i.e.,

$$P_h(x)(\equiv u_2(x)/u_1(x)) = c'_h(H^s(x)), \quad \forall x, \tag{14}$$

where $H^s(x)$ is the amount of housing constructed per unit land at x and $c'_h(H^s(x))$ is the marginal cost of construction at location x. It follows from (14) that $P_h(x)$ is indeed the housing price function. Observe that we can solve Eq. (14) to obtain $H^s(P_h(x))$.

3.1.2. Rent function

The rent at x, $R(x)$, is defined in (15) below as the difference between the revenue and the cost of construction per unit of land at x. Thus,

$$R(x) \overset{\text{def}}{=} P_h(x)H^s(P_h(x)) - c_h(H^s(P_h(x))), \quad \forall x. \tag{15}$$

The properties of the rent function are given in Appendix A. Note that even though in general housing price functions and rent functions are competitive equilibrium tools, they are well defined in this optimization model. These functions have the same properties as in an equilibrium model since housing and land have no external effects associated with them in the optimum.

Taking the integral of the rent function over the entire country yields ALR, the *aggregate land rent in the economy*, i.e.,

$$ALR = k\int_0^L R(x)dx. \tag{16}$$

Note that the right-hand side of the ALR equation consists of the aggregate land rents *in a complex* multiplied by the number of complexes in the economy.

[15] The same Pareto optimal solution can be obtained by a planner maximizing the social utility function which is obtained by adding all the utility functions of all the individuals in the economy; i.e., maximum $\int_0^{kL} u(Z(x), H(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)})dx$ subject to constraints (1)–(13) except for (5) which becomes redundant.

[16] If k is not an integer, there must be a fraction of a complex in the solution. Obviously, an actual allocation contains only complete complexes, which is the case for an integer k. Thus, in the optimal solution with a non-integer k, each complex is either smaller or larger than the optimal complex of the solution with an integer k, and the utility level for integer k is not higher than for real k. The distortion is negligible for a real but relatively large k. The problem of indivisibilities of economic entities is quite common in the economic literature (e.g. the indivisibility of the firm). In our case the problem might be more severe since k is likely to be relatively small. Thus, the subject of indivisibility of optimal complexes deserves a separate study.

3.1.3. The optimal 'budget constraint'

Let $j^i(x)$ be the index of the facility of club i to which a household residing at x travels. We define $Tr(x)$ as the *travel and usage expenditure of a household residing at* x, commuting to facilities $j^i(x)$ located at $x_{i,2j^i(x)}$, paying commuting costs $t_i(|x - x_{i,2j^i(x)}|)$ and *congestion tolls*, $c_2^i(j^i(x))$, for $i = 1,\ldots,I$. Thus,

$$Tr(x) \overset{\text{def}}{=} \sum_{i=1}^{I}\left[c_2^i(j^i(x)) + t_i(|x - x_{i,2j^i(x)}|)\right]. \tag{17}$$

Note that $Tr(x)$ is a continuous and differentiable function of x everywhere except at the facility locations, $x_{i,2j^i(x)}$, where $Tr(x)$ is continuous but not differentiable.

The following Eq. (18), the *household's optimal 'budget constraint'* at x, is a necessary condition for Pareto optimum.[17] The congestion tolls included in $Tr(x)$ are what distinguish the necessary condition below from an equilibrium budget constraint. We also define in (18) the *household's optimal expenditure function at* x, $e(p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)}, T_r(x), U)$, *in short* $e(x)$.

$$\begin{aligned} Y + v = {} & z(p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)}, U) + p_h(x)h(p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)}, U) \\ & + T_r(x) \overset{\text{def}}{=} e(p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)}, T_r(x), U), \quad \forall x. \end{aligned} \tag{18}$$

We can see that in (18), $p_h(x)$ indeed serves as the housing price, and the household's income $Y + v$ is independent of location and consists of the initial endowment of an individual household, Y, plus v, *an equal share of total alternative-shadow-land-rents* in the economy.[18] Thus, a household behaves in the optimum as a utility maximizer who considers as given: his income; the location $x_{i,2j}$ of all facilities (i,j); the quantities of CGs, G_{ij}, in these facilities; and the congestion tolls $c_2^i(j) \overset{\text{def}}{=} \frac{\partial c^i(G_{ij}, N_{ij})}{\partial N_{ij}}$ the household is required to pay when it uses facility i, j. Each club $i \in (1,\ldots,I)$ has m_i facilities spread throughout the complex and a household at x visits one facility of each club i.

3.2. Clubs

The external effects in the model are concentrated in clubs and therefore most of the equations in this section are not equilibrium relations.

3.2.1. Samuelson's rule

The necessary condition in (19) below determines G_{ij}, the optimal amount of CG for facility j in club i. The equation is a version of Samuelson's well-known rule about public goods.

$$\int_{x_{i,2j-1}}^{x_{i,2j+1}} \left[\frac{u_{i+2}}{u_1} n\right] dx = c_1^i(j), \quad \forall i, j, \tag{19}$$

where $c_1^i(j) = \frac{\partial c^i(j)}{\partial G_{ij}}$. On the right-hand side of (19) is the marginal rate of substitution in production between the CG and the composite good and on the left-hand side is the sum of the marginal rates of substitution in consumption of the users of facility i, j.[19]

3.2.2. Optimal facility location

The optimal facility location, $x_{i,2j}$, should satisfy the necessary condition in (20) below, which is also a necessary condition for the facility location to minimize aggregate transportation costs of patrons to facility (i,j).

$$\int_{x_{i,2j-1}}^{x_{i,2j}} n(x)t_i'(x_{i,2j} - x)dx = \int_{x_{i,2j}}^{x_{i,2j+1}} n(x)t_i'(x - x_{i,2j})dx, \quad \forall i, j. \tag{20}$$

In (20) the aggregate marginal transportation costs of patrons on one side of a facility equal the aggregate marginal transportation costs on the other side, so that a marginal shift in the facility location does not change aggregate transportation costs to the facility. It should be noted that linear t_i in (20) implies that on each side of the facility reside an equal number of patrons. The following lemma can now be proved:

Lemma 1. *A club's facility location is an interior point of the club's market area, and therefore of the complex. The market area of a facility is in a bounded segment of the complex.*

The proof of the first part of the lemma follows directly from (20) which requires that patrons should reside on both sides of the facility location. The proof of the second part of the lemma follows from the finiteness of the household's income which allows it to travel only a bounded distance.

3.3. Bid price functions and nodes

Bid price functions of housing and land are essentially tools of competitive equilibrium analysis. They can be employed in our optimization model since the land and housing markets are free from external effects. The bid price functions below are defined for given facility locations and the CGs in them, and for a given optimal utility level. The crucial assumption which allows bid function analysis is the assumption of a household's freedom to choose its location of residency which implies an equal utility level to identical households everywhere. This assumption is indeed part of this model as well as part of other urban competitive models. For a proof that bid housing price function analysis is compatible with the necessary conditions of this optimization model, see Appendixes 8.1.1 and 8.1.2 in the Web Appendixes. By nodes we refer to facility locations and boundaries.

3.3.1. Bid housing price functions

Let $Tr(x, j^1, \ldots, j^I)$ be the sum of the home-facility commuting costs plus the congestion tolls $c_2^i(j^i)$ a household residing at x pays when traveling to each of the I facilities, $j^1, \ldots, j^I$, as specified in (18), where j^i is the index of facility j of club i, i.e., $j^i \in (1,\ldots,m_i)$. The facility j^i is located at $x_{i,2j^i}$, with a given quantity of CG, G_{i,j^i}, i.e.,

$$Tr(x, j^1, \ldots, j^I) \overset{\text{def}}{=} \sum_{i=1}^{I}\left[c_2^i(j^i) + t_i(|x - x_{i,2j^i}|)\right], \tag{21}$$

$$\forall x, j \text{ and } i \text{ s.t.}, \; 0 \leqslant x \leqslant L, \quad j^i \in (1,\ldots,m_i), \quad i = 1,\ldots,I.$$

For the household to reside at x and travel to the given I facilities (j^i), the household's optimal budget constraint must fulfill,

$$Y + v = z(p_h(x), \left(G_{i,j^i}\right), U) + p_h(x)h(p_h(x), \left(G_{i,j^i}\right), U) + T_r(x, (j^i)). \tag{22}$$

where $(G_{i,j^i}) = (G_{1,j^1}, \ldots, G_{I,j^I}); (j^i) = (j^1, \ldots, j^I); z(p_h(x), (G_{i,j^i}), U)$ is the compensated demand function for the composite good Z, defined in (11) and $h(p_h(x), (G_{i,j^i}), U)$ is the compensated demand function for housing H, defined in (10).

The vector $\left(\left(G_{i,j^i}\right), (j^i), U\right)$ is fixed and given and so is the household's income $Y + v$. The only variable remaining to be determined

[17] Note that if $i = 1$ is a production club, then the expression $(-c_2^1)$ is the marginal product of labor which attains positive values and appears as income in the household's optimal budget constraint. In this case, the model has a non-zero solution even if Y vanishes.

[18] Namely, $v = \frac{LR_A}{N}$, where $R_A \geqslant 0$ and if $kL < \mathcal{L}$ then $R_A = 0$. See also (25), (26) and the discussion that follows at the end of Section 3.

[19] For club 1, the production club, after substituting $u_3 = 0$ in (19) reads $0 = c_1^1(j)$. To understand the meaning of (19) when $i = 1$, consider the production function $G_{1,j} - c^1(j)$ in perfect competition. Then the product and production factor are both the composite good whose price is 1. The profit maximization condition in competition is an equality between the value of the marginal product of the production factor $G_{1,j}$ and the product price, i.e., $\frac{\partial}{\partial G_{1,j}}(G_{1,j} - c^1(j)) = 1$ which implies that $0 = c_1^1(j)$, i.e., (19) for $i = 1$. Thus, in the case of the production club, the necessary condition (19) is simply the condition for profit maximization in perfect competition.

at a given location x is the price of housing, $p_h(x)$. By substituting out $p_h(x)$ from (22) we obtain the *bid housing-pice of a household residing at x and traveling to facilities at* $\left(x_{i,2j^i}\right)$ where the household uses the CGs, $\left(G_{i,j^i}\right)$. We designate this bid housing price function by $p_h^b(x,j^1,\dots,j^I)$. What distinguishes one bid housing price function from another is the set of facilities to which the household travels. Income and utility levels are the same for everybody everywhere and are known parameters as are the CGs and facility locations. Therefore, once the facilities' indices of a bid housing price function are known, all information is revealed. Each bid housing price function has a different set of I facilities. In each of two different sets of indices there is at least one facility that the other lacks. For some vectors (j^i), there may be locations x for which p_h, substituted out of (22), is negative. In such cases, we set the bid housing price equal to zero. We can now prove the following lemma.

Lemma 2. *The bid housing price function is a continuous function of the distance x and twice differentiable, with a positive second derivative everywhere except at the I facility locations* $(x_{i,2j^i})$ *where it is continuous but not differentiable.*[20]

A household at location x, by choosing to travel to facilities that yield the highest bid housing price is actually choosing to attain the utility level at location x by spending the least of all possible costs other than the cost of housing. Such behavior by all households leads to an efficient allocation. In competitive markets, a household at x travels to the facilities that yield the highest bid housing price at x, because he then can outbid others competing for housing at x. Accordingly, $p_h(x)$, the housing price function at x, fulfills

$$P_h(x) = \max_{j^1,\dots,j^I} p_h^b(x,j^1,\dots,j^I) = p_h^b(x,j^1(x),\dots,j^I(x)),$$
$$\forall x \text{ where } j^i \in (1,\dots,m_i), i = 1,\dots,I. \tag{23}$$

The vector of indexes of facilities $(j^1(x),\dots,j^I(x))$ to which a household residing in x travels to is merely the vector $(j^1,\dots,j^I)$ that maximizes $p_h^b(x,(j^i))$ in (23). Thus, the housing price function is the upper envelope curve of all bid housing price functions as defined in (23) and besides being the housing price function it also determines the facility locations to which a household at x travels.

3.3.2. Bid rent functions

We define the bid rent functions as

$$R^b(x,j^1,\dots,j^I) = p_h^b(x,j^1,\dots,j^I)H^s\left(p_h^b(x,j^1,\dots,j^I)\right) - C_h\left(H^s\left(p_h^b(x,j^1,\dots,j^I)\right)\right)$$

The bid rent is a monotonic increasing function of $p_h^b(x,j^1,\dots,j^I)$ and fulfills $R^b(p_h^b = 0) = 0$. Therefore, in most cases we can use either the bid rent function or the bid price function.

3.3.3. Boundaries and facility locations

In the optimal allocation a *node* x_b on the x-axis is *a boundary point between club-i market areas*, if there are points x_l and x_r, $x_l < x_b < x_r$, such that all residents living in (x_l,x_b) consume the ith CG in a facility to the left of x_b, and all residents in (x_b,x_r) consume the ith CG in a facility to the right of x_b.

Let x_b be a boundary point of clubs $i_1,\dots,i_K$, $1 \leqslant K \leqslant I$ and of them only (when $K = I$, x_b is the boundary of the complex). For brevity of notation we also designate by K the set $(i_k, k = 1,\dots,K)$ and by $I - K$, the set $((i_k \notin K)$ and $(i_k \in (1,\dots,I)))$. There is a point $x_l, x_l < x_b$, (x_l can be any point between x_b and the next boundary point to the left of x_b) that residents at every point $x, x_l < x < x_b$ use the I CGs at the same facilities. We designate these facilities by $j_o^1,\dots,j_o^I$, i.e., $j_o^i = j^i(x), x_l < x < x_b$. In the same way, there is a point $x_r, x_r > x_b$, where all residents in the segment $x_b < x < x_r$ use the I iCGs at the same facilities. In this segment, if $i \in K$, then $j_o^i + 1$ is the facility in which residents consume the ith CG and if $i \in I - K$, j_o^i is still the facility in which residents of x consume the ith CG. The necessary condition associated with the boundary x_b now follows:

$$p_h(x) = \max_{j^1,\dots,j^I} p_h^b(x,j^1,\dots,j^I) \begin{cases} = p_h^b\left(x,j_o^1,\dots,j_o^I\right), & \text{for } x, s.t.\, x_l < x \leqslant x_b, \\ = p_h^b\left(x,\left(j_o^i+1,\forall i \in K\right) \cup \left(j_o^i,\forall i \in (I-K)\right)\right), & \text{for } x, s.t.\, x_b < x \leqslant x_r \end{cases}$$
$$\text{and } p_h(x_b) = p_h^b\left(x_b,j_o^1,\dots,j_o^I\right) = p_h^b\left(x_b;\left(j_o^i+1,\forall i \in K\right) \cup \left(j_o^i,\forall i \in (I-K)\right)\right) \tag{24}$$

Eq. (24) states that the bid function $p_h^b\left(x;\left[j_o^i+1,\forall i \in K\right] \cup \left[j_o^i,\forall i \in (I-K)\right]\right)$ and the bid function $p_h^b\left(x;j_o^i,\dots,j_o^I\right)$ intersect at x_b and are equal to the housing price there. Hence, each of the two bid functions must coincide with the housing price function not just in x_b, but in a neighborhood of x_b as well. Note that the lowest two lines in (24) are the actual necessary condition. Below, x_b is indexed according to the rules set up in Section 2.

$$x_b = x_{i_1,2j_o^{i_1}+1} =,\dots,= x_{i_K,2j_o^{i_K}+1}.$$

For the proof that (24) is compatible with the necessary conditions, see Appendix 8.1.2 in the Web Appendixes.

The location of facility j of club i in our model is a node located at $x_{i,2j}$. The transportation cost function, $t_i(|x - x_{i,2j}|)$, is a continuous and differentiable function of x everywhere except at $x = x_{i,2j}$ where it is not differentiable. Since $Tr(x)$in (21) contains sums of transportation cost functions, it is continuous and twice differentiable everywhere except at facility locations where it is continuous but not differentiable. This property is passed on to p_h^b solved from (22) and (23) (see Lemma 2). Since $p_h(x)$, the housing price function, consists of segments of bid housing price functions that intersect at boundaries, it must be continuous and twice differentiable too except at facility locations and boundaries where it is continuous but not differentiable. To sum up the analysis, we write it in the form of a Proposition.

Proposition 1. *The housing price function,* $P_h(x)$*, is a continuous and twice differentiable function of x with a positive second derivative everywhere, except in nodes where it is continuous but not differentiable.*

Consecutive facilities of the same club may hold different quantities of the CG. Hence, households residing on different sides of a clubs' boundary may consume different quantities of one or more CGs (depending on whether the boundary is of one or more clubs and whether consecutive facilities have different quantities of their CG). With discontinuous changes in quantities of CGs consumed in consecutive facilities, discontinuous changes in households' consumption of housing and the composite good may be observed as well when crossing a clubs' boundary. In the following proposition we prove that this is not the case with housing and if two households live at the same location they consume the same amount of housing regardless of where they use the CG. This is stated formally in the following Proposition:

Proposition 2. *The household's housing consumption, H(x), is continuous everywhere, including in boundary and facility locations. Also continuous everywhere are the density of population, n(x), and the supply of housing,* $H^s(x)$.

[20] For a proof of Lemma 2 see Appendix A.

It should be noted that unlike the continuity of the supply and demand of housing, the household consumption of the composite good, as well as its CGs' consumption may be discontinuous in boundaries. For details and proof of the proposition, see Appendixes 8.1.1 and 8.1.2 in the Web Appendixes.

3.3.4. Market areas

In Section 2, we assumed that a market area served by a facility is a connected segment of the *x*-axis. Thus far we have used this assumption only for simplifying the notation. Now we prove this assumption endogenously in Lemma 3 for clubs with linear transportation cost functions.

Lemma 3. *The market area of a club with linear transportation cost function is a connected segment of the x-axis.*

For a proof, see Appendix 8.2.1 in the Web Appendixes and Fig. 5. Lemma 1 and Lemma 3 yield the next Proposition:

Proposition 3. *The market area of a club's facility is a bounded area and the facility is located in its interior. Market areas of clubs with a linear transportation cost function are compact.*

Recall that in this study we investigate only allocations in which market areas are connected.

3.4. The Henry George rule

The alternative land rent, R_A, is the land rent at the boundaries of a complex, i.e., R_A=$R(L)$. R_A is the lowest land rent anywhere in the complex. A necessary condition for Pareto optimum of an economy with identical households is the following relation:

$$v = \frac{R_A \mathscr{L}}{\mathscr{N}} \tag{25}$$

where v is the household's income from its share of alternative land rents (see also (18)). The Kunn-Tucker conditions imply that if $\mathsf{L} < \mathscr{L} \Rightarrow R_A = v = 0$ and when $\mathsf{L} = \mathscr{L} \Rightarrow v, R_A \geqslant 0$.

The last necessary condition for an optimum is the Henry George rule,

$$DLR \equiv \int_0^L (R(x) - R_A)dx = \sum_{i=1}^{I} \sum_{j=1}^{m_i} \left(c^i(j) - N_{ij}c_2^i(j)\right), \tag{26}$$

The term $\int_0^L (R(x) - R_A)dx > 0$, is the *differential land rents* (*DLR*). Since the DLR on the left-hand side of (26) is positive, so is the term on the right-hand side of the equation, i.e., the aggregate provision cost, $\sum_{i=1}^{I}\sum_{j=1}^{m_i} c^i(j)$, minus the aggregate congestion tolls, $\sum_{i=1}^{I}\sum_{j=1}^{m_i} N_{ij}c_2^i(j)$ (see also (17) and (18)). This means that congestion tolls cannot be the sole source of financing the clubs' operations. In (26) the DLR exactly equals the remaining deficit of the clubs after congestion tolls are paid to the clubs.[21] Therefore, the only net profits in the economy are the alternative land rents. It follows from (25) that in the optimum the overall profits in the economy, if any (i.e., if $R_A > 0$), are distributed among the general population.

4. Decentralization

In this section we deviate from the analysis of agglomeration to discuss briefly the issue of implementation of the optimal allocation. The problem is that on the one hand, in a laissez faire allocation each facility owner possesses market power and if left to his own devices he will engage in spatial monopolistic competition (of the type discussed in Beckmann (1999)), the outcome of which is usually inefficient. On the other hand, a full involvement of a local government in the production and distribution of CGs is likely to become inefficiently managed. One reason for this possible mismanagement is that no real and clear cut goal faces a government-appointed manager of a facility. Another reason is the lack of incentive of such a manager to operate the facility efficiently.

A local government (of a complex) possesses a better option for its clubs' management than either complete lack of intervention and allowing a regeme of laissez faire or by full government intervention. This option being the decentralization of the optimal allocation by a local government partial intervention in local market operations that is limited to the use of only transfer payments (i.e., taxes and subsidies) and spatial regulations (e.g., zoning). The purpose of this decentralization is that the city attains the desired efficient allocation or comes close to it. The general theory of decentralization follows from "The Second Fundamental Theorem of Welfare Theory" (e.g., see Mas-Colell et al. (1995), Chapter 16, Proposition 16.D.1) which proves that it is possible in general to decentralize a Pareto optimal allocation. It is shown in Mas-Colell et al. (1995) that every Pareto optimal allocation (x^*, y^*) (his notation) has a price vector $p = (p_1, \ldots, p_L) \neq 0$, such that (x^*, y^*, p) is a price quasi-equilibrium with transfers. In other words, in a sufficiently well-behaved economy with price-taking agents, prices and income transfers exist that yield the optimal solution as a market allocation. Furthermore, in Hochman and Ofek (1979) it is argued that by performing decentralization a local government can improve its tax-base and income. Actually, the local government knows that an action it takes is in the right direction and that its policy is correct if in return, the city's land rents plus government net (i.e., the increase in tax income minus expenditure on the action taken) income increase.

In the case of non-spatial clubs, an efficient equilibrium exists that does not require any government intervention (e.g., see the outset in Hochman et al. (1995)). However, in the case of spatial clubs, a government intervention is needed to instigate the provision of optimal quantities of CGs at the optimal nodes. In Hochman et al. (1995) we have concluded that decentralization is impossible and a full government involvement is needed. Below, however, it is shown that decentralization is possible and is essentially implemented by local governments in real-life situations.

We first investigate cases in which club operators can locate facilities only in predetermined sites matching the optimal facility locations. We discuss this restriction and partially relax it later on. There is no unique way to decentralize our optimum and for different clubs, different methods may be suitable. A natural way to decentralize our optimum is to allow each facility operator to charge each user the congestion toll $c_2^i(j)$, which ensures the fulfillment of (18). The facility's income from user-charges is $N_{ij}c_2^i(j)$ and, in general, this toll is not sufficient to cover the full cost of running an optimal facility. A facility's loss is $c^i(j) - N_{ij}c_2^i(j) > 0$ and the local government has to provide the missing funds to cover it.[22] The Henry George Rule (26) ensures that the differential land rents, which are taxable by the local government, are sufficient to cover the total deficit.

The above decentralization method, in which facility operators charge patrons with congestion tolls and are subsidized by the

[21] In the case of the industrial club, the term $(-N_{1,j}c_2^1(j) > 0)$ is the wages paid to the workers in the facility and $(-c^1(j)) > 0$ is the value added over the value of the input of the composite good, G_{1j}. Therefore, $c^1(j) - N_{1j}c_2^1(j) > 0$ is the deficit of the production club's facility. Thus, each facility has to receive a subsidy from the local government that can be financed by an optimal taxation of land rents. This result is well-known in the literature.

[22] Not all club facilities in a complex must suffer losses and some of them may have profits. However, the overall combined costs of all the clubs in the complex are always higher than the overall congestion tolls. To see that consider the following Henry George rule (see also (26)), $0 < DLR \equiv \int_0^L (R(x) - R_A)dx = \sum_{i=1}^{I}\sum_{j=1}^{m_i} \left(c^i(j) - N_{ij}c_2^i(j)\right)$. The double summation in the equation is positive although it may contain some individual negative terms of facilities whose congestion tolls exceed their expenses. Such facilities need to be taxed instead of subsidized.

local government, suffers from the fact that facility operators have little incentive to behave efficiently. By doing nothing and acquiring a government subsidy, a facility operator obtains the subsidy as positive profits, while by behaving optimally the best he can do is end up without losses (see Hochman et al. (1995)). Another problem with this method is that the government may have a knowledge about the combined performence and needs of all the clubs in the complex but it lacks the knowledge of the needs of individual clubs and how to divide the taxed DLR between them.

Despite these drawbacks there are circumstances in which the decentralization by subsidizing a facility can be used. Consider, for example, the case in which the provision costs are divided into costs of constructing a facility (fixed costs) and non-decreasing marginal costs of operations. In such a case, the government can construct the facility, thus paying the fixed costs, and then lease the facility to a private operator who is allowed to charge users the marginal cost while maintaining current operations and paying the variable costs. Knight (1924) showed that for a road system there are circumstances in which profit maximizing user-charges are equal to optimal congestion tolls. Indeed, if the facility operator incurs positive profits, the government can obtain these profits as lease payments and redistribute them back to households. Most toll roads currently operate according to this principle.

Another decentralization method is applicable in general to all cases including those in which division to fixed and non-decreasing marginal costs are not relevant. We let an asterisk designate optimal values of variables and $p^d_{G_{ij}}$ be the price a household pays per unit of G_{ij} it consumes at facility (i,j), where $p^d_{G_{ij}} \overset{\text{def}}{\equiv} \frac{c^i_1\left(G^*_{ij},N^*_{ij}\right)}{N^*_{ij}}$.[23] Let the price, $p_{G_{ij}}$, be the price the facility ij operator receives per unit of CG he provides, which is $p_{G_{ij}} \overset{\text{def}}{\equiv} N^*_{ij} p^d_{G_{ij}} = c^i_1\left(G^*_{ij}, N^*_{ij}\right)$. All agents are now price takers since they have to accept the government's dictated price. Under this condition, G^*_{ij} is the amount of CG that maximizes the facility operator's profits and N^*_{ij} is the number of patrons that visit the facility. Note that in this case the facility has positive profits since, $p_{G_{ij}G^*_{ij}} = c^i_1\left(G^*_{ij}, N^*_{ij}\right)G^*_{ij} > c^i\left(G^*_{ij}, N^*_{ij}\right)$, where both c^i_1 and c^i_{11} are positive (see Section 2). Finally, let $S(x)$ be a government subsidy to a household located at x, $S(x) \overset{\text{def}}{\equiv} \sum_i \left[\frac{c^i_1\left(G^*_{i,j^i(x)},N^*_{i,j^i(x)}\right)}{N^*_{i,j^i(x)}} G^*_{i,j^i(x)} - c^i_2\left(G^*_{i,j^i(x)}, N^*_{i,j^i(x)}\right)\right]$ where the summation is over all the clubs which a household at x uses. This subsidy compensates households for those charges which are higher than the congestion tolls. The government can finance this subsidy by taxing the facilities' profits and differential land rents. We can now prove the following Proposition,

Proposition 4. *The price vector* $(p^d_{G_{ij}}, p_{G_{ij}}, p^*_h(x))$, *the household's subsidy function S(x), the model setup in* Section 2 *for a given basic configuration and the optimal facility locations constitute a price quasi-equilibrium with transfers that yield the model's Pareto optimal allocation.*

For the proof, see Appendix 8.3.1 in the Web Appendixes. Note that in this decentralization method, facility operators are price takers and customers pursue the least expensive facility which fulfills their needs.

If all facilities of a club are the same, i.e., they all have the same number of patrons and the same amount of CG, the subsidies to a household are identical everywhere. However, if there are clubs with three or more facilities in a complex, some of them may have different number of patrons than others. In this case the subsidies required to compensate households become location-dependent and may be different in different neighborhoods. In practice, local governments do not bother to return income that they tax from club to the particular users of the clubs and instead they add this tax income to the general municipal income and reduce the tax burden of the general population. See Hochman and Ofek (1979) for a theory on actual local government behavior.[24] Changing tax bases may cause deadweight losses (see Hochman (1990)), however, these inefficiencies might still be the best option.

So far we have assumed that club managers face predetermined facility locations in a complex, which, to the most extent, resembles real life. Club sizes and locations are detailed in city master plans, their number is regulated and each club requires a permit. As such, no decentralization of the choice of facilities locations (i.e., creating conditions in which club operators will choose the efficient facility locations) is really required. The fact that in real-life decentralization of the choice of facility locations does not take place is a clear indication of the complexity of such a process.

In what follows, however, we investigate the decentralization of locating facilities, since it may be applicable to some clubs that deliver home their CG (e.g., the electricity and water utilities, fire fighting). The optimal facility location is the one that minimizes overall commuting costs from the market area, i.e., (20) has to be fulfilled. If households are left to pay for their own commuting, the facility operators will choose facility locations that maximize their patronage and profits and disregard the effect the facility location has on commuting costs. This may lead operators to locate their facilities inefficiently. For example, if in a complex there are two facilities of the same club, both of them will locate in the center of the complex, each trying to add to its market area the more densely populated areas in the center of the complex while relinquishing sparsely populated areas closer to the complex boundary. To induce facility managers to locate efficiently, their goal function should include the minimization of their patrons' total commuting costs, so that (20) is satisfied. To achieve this goal, each facility operator should transport his patrons by himself, in return for a predetermined lump sum payment. The lump sum should be the same to all residents living on the same side of the facility and equal to the commuting costs of an individual living at the boundary of the market area. With this method of payment, a facility manager has an incentive to choose a facility location that minimizes overall transportation costs, since then he will be maximizing his profits from transportation. Indeed, a first-order condition for such a minimization is (20). At the same time, the local government should tax the additional profits of the facility owner and redistribute them among the club's patrons so that the lump sum transportation payment of a household minus the transport subsidy it receives equals the household's actual transportation costs. In this case, the redistributed amounts vary from one location to another even within the market area of the same facility and even if all facilities are the same. Besides impossible income redistribution and the unrealistic assumption of price-taking agents, which by themselves render this type of

[23] The price $p^d_{G_{ij}}$ is not equal to the household's marginal rate of substitution between the CG and the composite good. It is, therefore, not really a (Lindahl) price but more a lumpsum tax. However, a price-taking individual will consume the correct optimal CG since this is the quantity provided by the closest facility and it is the better option of CG consumption compared to other facilities of the same type. This lumpsum is preferred over Lindhal pricing since it is the same for all users of a facility.

[24] Retail stores are facilities of a club that have another method of financing its operations. Stores provide the service of distributing consumption goods to the general public. They buy goods from producers at gross prices and sell them at higher retail prices. Stores differ from each other in the type of goods they sell, the quality of products and services, prices, accessibility, etc. In practice, although retail stores are very competitive, they do not behave as price-takers and the method by which customers pay for their services is not applicable to Proposition 4. Yet the allocation of stores could be optimal if the government would tax stores profits and refund buyers for excess payment. In real life, taxes on stores are high but the direct refunding of buyers is practically impossible and the income from these becomes part of the government's general budget.

decentralization unpractical, it also suffers from the inherent problem that the facility operator can only acquire monetary travel costs. Costs involving the value of travel time must be borne by the individuals themselves and since in the decentralized method the facility operator only minimizes monitary commuting costs, he does not locate the facility optimally. In view of these shortcomings of this decentralized method, it would seem that the determination of potential facility locations should be left to city planners.

Nonetheless, in clubs where the CG is delivered home like utilities, no travel time of patrons is involved and the above method of choosing facility locations can be used in practise. If this decentralized method is used then the added delivery costs that households residing closer to the facility pay should be returned to the payees in one way or another. Otherwise, if all households regardless of their residential location pay the same price for the delivered CGs (e.g., water, electricity) than the incentive of households to concentrate around the facilities will be impaired.

The last remaining issue of decentralization is how can a local government know which of its actions actually improve efficiency and which reduce it. Furthermore, what is the incentive of the local government to act efficiently? The answer is that any action of taxation, regulation and provision of collective goods and services improve efficiency if total government net (minus expenses) income plus the differential land rents in the city increase due to the action. This is also the incentive of the local government to be efficient, since total resources available to the municipality (tax-base) then increase. For further details see Hochman and Ofek (1979). It should be noted that only the basic industry in a city, which we termed here as 'the production club', should be subsidized by the local government (see footnote 12 above as well as Hochman (1981, 1990, 1997)). All other enterprises that provide goods and services to the local population (termed here 'clubs') constitute sources of income to the municipality.

5. Agglomeration of spatial clubs and concentration of households

In this section we investigate agglomerations of spatial clubs and concentration of households in optimal allocations. We first describe a few general characteristics of the optimal solution and discuss briefly general effects of congestion and commuting costs. Then we elaborate on allocations of two simple basic configurations, each of which characterizes a particular type of club agglomeration. The first deals with perfect agglomeration of facilities of different clubs and the second involves imperfect agglomeration of facilities. We give an example in which perfect agglomeration of facilities in the center of a complex is a global optimum. We then extend these results and show that these two types of agglomerations apply to other configurations as well. We conclude the section by showing that a variety of global optima can be attained.

As a reminder, a complex configuration is a vector with I integer components m_i whose GCD is one. Each m_i designates the number of facilities of club i in a complex. The variable k measures the number of complexes in the economy.

5.1. General characteristics

In a ring-shaped economy that is partially unoccupied, even if we assume that the occupied land constitutes a single connected segment $(O, \mathsf{L}), O < \mathsf{L}(= kL) < \mathscr{L}$, and all the unoccupied land is the segment $(\mathsf{L}, \mathscr{L})$, there are two edges to the occupied land: $\mathsf{L}(\equiv kL)$ and $O(\equiv \mathscr{L})$.[25] Since all CGs are essential, these two edges must be boundary points to all clubs, i.e., the origin, O, is the left boundary of the first market area of each club and L is the right boundary point of the last market area of each club.

We first show that an agglomeration of clubs in an economy with a uniform population distribution is *ineffective*. In an edgeless economy an agglomeration is ineffective if besides the allocation with the agglomeration there is also an infinite number of other allocations, all equivalent to the one with the agglomeration, i.e., they all have the same utility level and the same consumption baskets as the allocation with the agglomeration, but all of them are without an agglomeration of facilities. An ineffective agglomeration in an economy *with* edges is an agglomeration in an allocation, possibly a unique and optimal one, that becomes an ineffetive agglomeration when the economy is turned into an edgeless one.

Consider the following example of an allocation of I clubs in an economy with two edges (i.e., $O < \mathsf{L}(= kL) < \mathscr{L}$) and a homogenous population distribution. All the different clubs in the economy have the same number of identical market areas and each club's facility is located in the center of the club's market area, i.e., $m_i = 1$, $\forall i$, and the number of complexes, k, is also the total number of facilities of each club. Since the extreme two boundaries of every club coincide, it follows that all market areas are common to all clubs and the facilities of all I clubs are jointly located in the center of each of the joint market areas. In other words, facilities of all clubs agglomerate in a single location at the center of each complex.

In this example, since the population is uniformly distributed over space, all households consume the same amount of housing and the quantity of CG in each of the facilities of a club must be the same for the utility of all households to be identical. Suppose that all households have the same resources and utility level, hence, all households must also consume the same amount of composite good in order to have the same utility level. In short, in the economy just constructed, all households have the same utility level and consume identical bundles of housing, composite good and CGs. In addition, market areas are common to all I clubs and in the center of each market area facilities of all I clubs are agglomerated.

Suppose now that the unocupied segment in the economy is eliminated so that $kL(\equiv \mathsf{L}) = O(\equiv \mathscr{L})$. Then the economy no longer has edges. In this edgeless economy, the previous allocation of clubs with common market areas and agglomerations of facilities of the I clubs still exists, but the last boundary of the last market area of each club coincides with the first boundary of the first market area of each club as well as with the first and last boundaries of all other clubs. However, in this edgeless economy, unlike the economy with edges, there are no points that *must* be a boundary to all types of clubs (that the edges of the economy were). Actually, a club in the edgeless economy is free to have its boundaries anywhere as long as the distance between two consecutive boundaries of the same club are constant and equal to L. Therefore, all clubs can be arbitrarily arranged in a consecutive order and the location of boundaries and hence of facilities of different clubs, can be arranged so that the distance of a facility of one club from the next consecutive club's facility is L/I. The sizes of a club market areas remain unchanged as in the allocation with agglomerations, the location of each facility remains in the middle of its market area and the quantity of CG in each facility remains as is. Furthermore, households remain where they are so that the population density remains uniform. The result of such an allocation is, first of all, that it has no agglomeration of facilities; in fact, the facilities are distributed evenly throughout the ring. Secondly, since the market areas are the same in the two allocations and the distribution of population is uniform, the number of patrons and travel distances in each market area

[25] If the occupied land is not connected, there are more than just two edges to the economy, a fact that strengthens our arguments.

remain the same as in the economy with edges. Consequently, total commuting costs in each facility of each club are unchanged as well as total provision costs. It follows that each household consumes the same basket as before and therefore has the same utility level, but this time there is no agglomeration of facilities. As a matter of fact, in the edgeless economy there is an infinite number of allocations with the same utility level and all without an agglomeration of facilities.[26]

The above example implies that an allocation with an agglomeration of facilities of different clubs in an edgeless economy constrained to a uniform population distribution is just one of an infinite number of equivalent allocations, all with the same consumption bundle and utility level but without an agglomeration of clubs. This, in turn, implies that the agglomeration of facilities of different clubs in an economy with a uniform population distribution do not contribute to welfare and is therefore, an *ineffective* agglomeration. The fact that in the above example of an economy with edges there is a unique optimal allocation and that facilities agglomerate in it is entirely due to the economy's edges and to the technical coincidence that all clubs have market areas of the same size.

Therefore, in order to avoid confounding the main issues of this paper and to concentrate on essentials, from here on we restrict our analysis to solutions of the model that satisfy the following Condition A:

Condition A. In an optimal allocation investigated here:

(i) The number of complexes, k, is an integer.
(ii) There is no vacant land in the economy, i.e., $\mathsf{L}(=kL)=\mathscr{L}$ and $R_A > 0$, where R_A is the shadow rent at a complex boundary.

Part (i) of Condition A is intended to avoid the problem of indivisibility of optimal complexes by dealing only with population sizes that are integer multiplications of an optimal complex size. In that we follow Hochman et al. (1995). Part (ii) is intended to achieve an edgeless economy to avoid the 'edge-of-the-economy' effect. Under Condition A, for every area $\mathscr{L}$ of the economy we have a lower bound of $\mathscr{N}, \underline{\mathscr{N}}(\mathscr{L})$, such that every $\mathscr{N}$ fulfilling Condition A, also fulfills $\mathscr{N} > \underline{\mathscr{N}}(\mathscr{L})$. Then, $\mathsf{L}(\equiv kL)=\mathscr{L}$.

We attribute the term central location pattern (CLP) to a club's location pattern in which every market area is common to all clubs, and facilities of all clubs are located in the center of the joint market area. Thus, in the above example, the initial location pattern with agglomeration of facilities of all I clubs is a CLP.

We now introduce a new virtual tool, which we term a 'rotation' of a club. This tool is useful in an edgeless, ring-shaped, uniform density economy and we use it in the proof of the next Proposition.

Definition. Let a rotation of club i be a shift to the right of all the nodes of club i by the same distance while keeping the population and the nodes of the rest of the clubs unmoved.

Club i nodes are all the boundaries and facility locations of club i and they all shift in a rotation of club i. The locations of nodes of clubs other than i remain constant in a rotation of club i. as do the quantities of CGs in the facilities of all clubs, including those of club i. This rotation maintains constant distances between club i nodes and keeps the locations of households unchanged.

We now return to the first-best allocation to prove the following proposition:

Proposition 5. *In a first-best allocation of a club economy the population density is never uniform, i.e., there are segments of the economy in which* $n(x)\neq\mathscr{N}/\mathscr{L}$.[27]

We prove the proposition by showing that a contradiction occurs when assuming a homogeneous population distribution in an optimal allocation. This contradiction is obtained when clubs are rotated so that all of them have a facility at the same agglomeration point. At this agglomeration point, the density of population must be at its maximum and decline gradually as the distance to the agglomeration point increases. This contradicts the homogeneity assumption. For a detailed proof, see Appendix 8.4.1 in the Web Appendixes.

Another property of an optimal solution in a spatial club economy is that to a set of necessary conditions there is always a symmetric allocation that fulfills these conditions, where the symmetry is with respect to the center of the complex.[28] Each configuration has a spatial *symmetric structure* of its own. In a symmetric structure, if m_i is an odd number of club i facilities, the club has a facility $j, j=\frac{m_i+1}{2}$ that is located in the middle of the complex, with its market area spread symmetrically around the facility. The remaining m_i-1 facilities of club i, where, m_i-1 is an even number, are arranged consecutively and are located symmetrically with respect to the center of the complex, so that each facility has its mirror image facility on the other side of the center. Thus, facilities j and j' are two facilities that are symmetric to each other if $j+j'-1=m_i$.

When m_i is an even number there is no facility in the center and instead a boundary is located there. In this case, all the facilities are symmetrically located around the center so that a facility j is the mirror image of its symmetric facility j' on the other side of the center and $j+j'-1=m_i$. The population density is also symmetric around the center of the complex. It should be recalled that in each basic complex configuration there is at least one club with an odd m_i; otherwise the configuration would have a common divisor greater than one and would not be basic. Therefore, there is at least one facility in the center of each complex. In Proposition 6 below we prove that for a set of necessary conditions there is a symmetric optimal solution of the complex.

Proposition 6. *Given that the model's functions fulfill the conditions in* Section 2, *that the model's configuration is given and that Condition A is satisfied, then there is always an optimal complex with a symmetric structure (as described above) that solves the model.*

The intuition behind the above proposition is that if an allocation is optimal on half of the complex area, its mirror image must be optimal on the other side of the center since commuting is a function of only the distance and is therefore symmetric. A detailed proof is in Appendix 8.4.2 in the Web Appendixes.

In this section we presented Propositions 5 and 6, which state that a first-best allocation has a population distribution that is

[26] If in the model the quantity of housing consumed is fixed and not included in the utility function (e.g., as in Mohring's early urban model) then the outcome of utility maximization is that the population is distributed uniformly and the resulting allocation is an efficient competitive equilibrium. An agglomeration of facilities in such an allocation is therefore efficient, competitive and ineffective. When the allocation is second best because it is constrained to a uniform population distribution and housing is in the utility function, then an agglomeration of facilities is both ineffective and inefficient.

[27] When there is at least one transportation cost function whose second derivative is strictly negative, i.e., there is at least one i_o s.t., $t''_{i_0}<0$, we can strengthen the proposition's result. Actually, if $t''_{i_0}<0$, there is no segment in the economy in which the density of population is constant. To see this, consider (B2) in Appendix A, in which we see that when $\dot{p}_h$ vanishes at a point x_o (i.e., the housing price function is constant at x_o), $\ddot{p}_h(x_o)$ is positive. This, in turn, implies that in a segment $(x_o, x_o+\varepsilon)$, where ε is small and positive, $\dot{p}_h>0$. Thus, if $\dot{p}_h=0$ at any point, it is positive immediatly after that point. Because $\dot{p}_h=0$ if and only if the gradient of the density function, $\dot{n}(x)$, is zero as well, the assertion follows. ■

[28] However, we do not show that this solution is unique or even that non-symmetric optimal allocations do not exist.

never uniform and that a first best always has a symmetric allocation that solves it. In the next section, we discuss the effects of congestion and transportation costs on the solution to the model.

5.1.1. Congestion and commuting effects

We discuss here briefly the effects of congestion and transportation costs on the efficient way to provide a CG to households. We first note that if there are no scale economies in the provision cost function of a CG, i.e., $c(G,N) = Nc(G,1)$, then the CG is a private good to be provided to each household at home and not in a facility. The cost of providing the CG at home is $Nc(G,1)$ and it is equal to $c(G,N)$, which is the cost of providing the CG at a facility. However, in a facility there are also commuting costs, which are saved by providing the CG at home. If the CG is a pure public good, i.e., $c(G,N) = c(G,1)$, then commuting costs are the only reason for having more than one facility for the distribution of the CG. Because an added household to a facility's existing patrons must reside farther from the facility than all the other patrons, the accumulated commuting costs to a facility are a function increasing at an increasing rate of the facility's patronage.[29] Thus, although direct provision costs are not affected by the number of households using the facility, the overall costs of providing the CG, which include transportation costs, are increasing at an increasing rate, so much so that at a certain distance from a facility it pays to construct an additional facility rather than to increase the existing one. Actually, sufficiently high commuting costs relative to direct provision costs can make it worthwhile to supply even a pure LPG at home rather than at a facility.

From the above discussion, it follows that both increasing congestion in a facility and/or increasing commuting costs to it reduce patronage in each of the two cases and increase the number of facilities in a complex. There are, however, some differences between these two effects. For example, a high degree of congestability and relatively low commuting costs will yield facilities with similar sizes in both densely populated areas and sparsely populated areas. The difference between these two types of areas is the number of facilities per unit land that is higher in denser areas. Alternatively, when the degree of congestion is low and transportation costs are high, facilities' patronage in densely populated areas is much higher than in sparsely populated areas but the number of facilities per unit land is not so much different.

The congestion and transportation costs discussed above affect mainly the global optimum solution and when the configuration is fixed they affect only the size of the complex. Accordingly, if the CGs are highly congestable or transportation costs are high, then the whole complex becomes relatively small. Conversely, low congestion and low transportation costs yield a larger complex size.

In the following sections, we further characterize symmetric solutions of the model, essentially by identifying the relative locations of facilities and by typifying the housing price function and with it the density function. The question we face is whether agglomerations of facilities of various clubs actually take place in an optimal allocation. To answer this question, we characterize two optimal symmetric allocations, each of which having a different, simple configuration. We show that the concentration of households and agglomeration of facilities occur in an edgeless economy.

5.2. Perfect agglomeration

We first characterize the solution of the model with the configuration $\overbrace{(1,\ldots,1)}^{I}$. This solution results in a type of facility agglomeration that we term *perfect*. A second type of facility agglomeration that we encounter in the next section, we term *imperfect*. It turns out that these two types exhaust all types of agglomerations and can help us characterize solutions of the model in general, as we do in Section 5.4.

The term *perfect agglomeration* refers to an agglomeration of facilities of different clubs located at the same point.[30] An allocation with a central location pattern (CLP) in which all the facilities in a complex are located in the center of the complex is an example of perfect agglomeration. In what follows, we show that our model with the configuration $\overbrace{(1,\ldots,1)}^{I}$ has an optimal solution with a CLP that satisfies the necessary conditions specified in Section 3. From Proposition 6 we know that all the facilities in a CLP allocation are located in the center of a symmetric complex. We designate the optimal values of variables of the model with the configuration $\overbrace{(1,\ldots,1)}^{I}$ by the superscript c. In the Proposition below we investigate properties of the model's solution.

Proposition 7. *An optimal allocation of a spatial club economy that satisfies Condition A with the configuration* $\overbrace{(1,\ldots,1)}^{I}$ *consists of* k *symmetric complexes. Each of these complexes has a CLP in which facilities of all clubs are located in the center and the population is distributed symmetrically around the complex center. The population density function and the price of housing function are both symmetrical around the complex's center, and are continuous and differentiable everywhere except at the nodes where both functions are continuous but not differentiable. Both the density function and the housing price function are declining with the distance from the center and the housing price function has also a positive second derivative.*

For a detailed proof, see Appendix 8.4.3 in the Web Appendices. In the proof we show that there is an allocation with a CLP, a density function and a housing price function that satisfy the necessary conditions for an optimum and possess the properties mentioned in the Proposition above.

Corollary 8. *In an optimal allocation with a CLP, the agglomeration of facilities of different clubs in the center of each complex is accompanied by a concentration of households around the center.*

The proof follows directly from Proposition 7.

The housing price function of a CLP in an optimal complex is depicted in Fig. 3.

Definition. An optimal allocation of our model (set up in Section 2) with a given set of functions and the basic configuration M is a global optimum if any optimal allocation of the model with the same functions but with a basic configuration other than M, has a lower utility level.

In the example below we present a set of functions that satisfies additional specifications to the model's general functions introduced in Section 2. In this more specific set of functions the allocation with the CLP of Proposition 7 together with the complex configuration $(1,\ldots,1)$ is the global optimum solution of our model whose functions fulfill the additional specifications in the example below.

[29] Note that the positive first-order derivative of the transportation cost function with respect to distance is responsible for the second order derivative of the accumulated transportation costs in a facility as a function of the facility's patronage being positive.

[30] In a model where facilities occupy space, perfect agglomeration means that the areas occupied by the facilities are adjacent to each other with no households in between them.

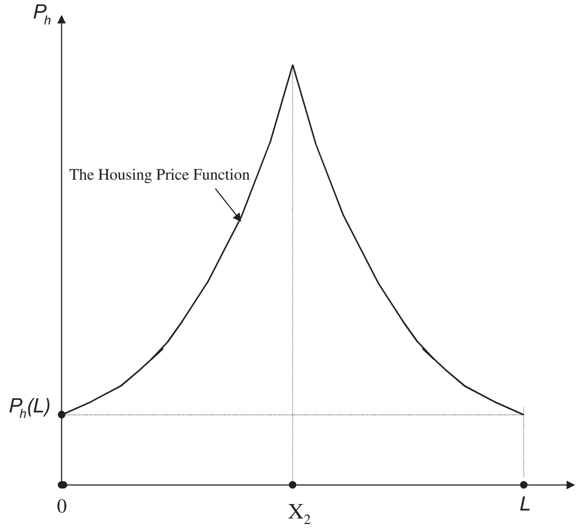

Fig. 3. The housing price function of an optimal complex with the configuration (1,...,1).

Example 1. Function specifications for a global perfect agglomeration in a CLP

Consider a model of an economy with spatial clubs, which in addition to the conditions on the functions set in Section 2, satisfies the following more specific conditions:

1. The utility function is of the form $u = U(H,Z,\psi(G_1,\ldots,G_I))$, where $\psi(G_1,\ldots,G_I)$ is invariant for permutations of the set $(G_1,\ldots,G_I)$, e.g., $\psi(G_1,\ldots,G_I)=\prod_{i=1}^{I}G_i$.
2. All clubs share the same transportation cost function, i.e., $t_i(y) = t(y)$, $\forall i$, and the same provision cost function, i.e., $c^i(G,N) = c(G,N)$, $\forall G, N$.

Lemma 4. *An optimal CLP allocation as described in Proposition 7 is a global optimum solution to the model with functions from the above example.*

The Lemma's proof is intuitively straightforward; for a formal proof, see Appendix 8.4.4 in the Web Appendixes.

It should be noted that a marginal change in the number of facilities in a complex is impossible and the smallest change is of one more (or less) facility. Therefore, sufficiently small variations in the specifications of the functions would leave the basic configuration of the global optimum intact. For example, if instead of using the utility function $u = H\cdot Z\cdot\prod_{i=1}^{I}G_i$ in the example, we would use the utility function $u = H\cdot Z\cdot\prod_{i=1}^{I}G_i^{1+\alpha_i}$, where $|\alpha_i|$ are sufficiently small yet different from each other, the global optimum allocation would still be a CLP. The same is true for small variations in the transportation cost functions of the different clubs or small differences in their provision cost functions. However, while the basic configuration of the global optimum may not change due to small variations, all other variables change continuously.

5.3. Imperfect agglomeration

Here we characterize the optimal allocation of the model with the basic configuration given below:

$$(m_i = 1, \forall i = 1,\ldots,I_1 \cup m_i = 2, \forall i = I_1+1,\ldots,I);\quad 1 \leqslant I_1 < I < \infty. \tag{27}$$

Perfect agglomeration, which has been investigated in the previous Section, is the agglomeration of facilities of different clubs in a single location (in the CLP the agglomeration of facilities at the center of the complex is of all the clubs). In imperfect agglomeration that is investigated here, facilities of different clubs share the same market area and agglomerate in a cluster in which facilities are close to each other but they are not necessaryly located at the same point. Thus, by imperfect we mean that the clusters may contain dwellings between the facilities. In addition, each cluster of facilities in a market area gravitates towards the center of the complex but steers clear of it. We say the cluster gravitates when it is closer to the boundary of its market area nearest the center of the complex than to the other boundary.

We first introduce the symmetric structure of the allocation with the configuration given in (27) as specified in Proposition 6. The symmetric structure possesses the following properties: (1) Each of the clubs $i \in 1,\ldots,I_1$, (henceforth SF clubs) have one facility located in the middle of the complex and its market area is the whole complex, and (2) the two facilities of each of the clubs $i \in (I_1+1,\ldots,I)$ (henceforth DF clubs), are symmetrically located on each side of the center of the complex and each of their market areas is extended between a complex boundary and the center. Altogether, the complex has I_1 facilities of SF clubs and $2(I-I_1)$ facilities of DF clubs, of which $(I-I_1)$ are sitting on each side of the complex center. The properties of the allocation with the configuration (27) are described by the series of lemmas presented in the rest of this section. Since all complexes are the same, we deal with a single representative complex that occupies the segment (O,L), namely, the first complex after the origin in the clockwise direction.

Lemma 5. *In the optimal allocation of the model with the configuration (27) discussed above, all the facility locations of the DF clubs are in the second and third quarters of the complex length.*[31] *The average density of the population residing between the two facilities of the DF club that is farthest from the center (one facility to the left and one to the right of the center) is higher than it is between these two DF facilities and the boundaries.*

For the proof of the Lemma, see Appendix 8.4.5 in the Web Appendixes.

In what follows, we show that all the DF clubs agglomerate in two clusters, one in the second quarter of the complex and the other in the third. Let $\bar{x}_2$ designate the facility location of the DF club in $(0,L/2)$, which is located closest to $L/2$, and let $\underline{x}_2$ be the location of the closest facility to the origin. In the Lemma above, we proved that $L/4 < \underline{x}_2 \leqslant \bar{x}_2 < L/2$. It follows that all the facilities of the DF clubs in the first half of the complex are located between $\bar{x}_2$ ($<L/2$)and $\underline{x}_2$ ($>L/4$) and are clustered together closely in the second quarter of the complex (and consequently, the DF clubs in the second half of the complex are clustered in the third quarter of the complex). We term such close groupings of facility locations a *cluster of DF facilities*. In the lemma above, we showed that such clusters of DF clubs are located closer to the center of the complex than to the boundaries. In such cases, we say that the DF clusters *gravitate towards the center of the complex.*

To clarify the role of transportation costs in an imperfect agglomeration of DF clubs, consider the following Lemma:

Lemma 6. *In an allocation with the basic configuration specified in (27), different DF clubs with proportional transportation cost functions share the same facility locations.*[32]

[31] By the term "quarter" we refer to a segment which results from a division of the complex's length into four equal consecutive segments. The first quarter is the segment farthest to the left and the other three quarters are numbered consecutively in the clockwise direction.

[32] Recall that in our model if two facilities of different club types share the same location, it means that they are adjacent to each other with no residential area between them.

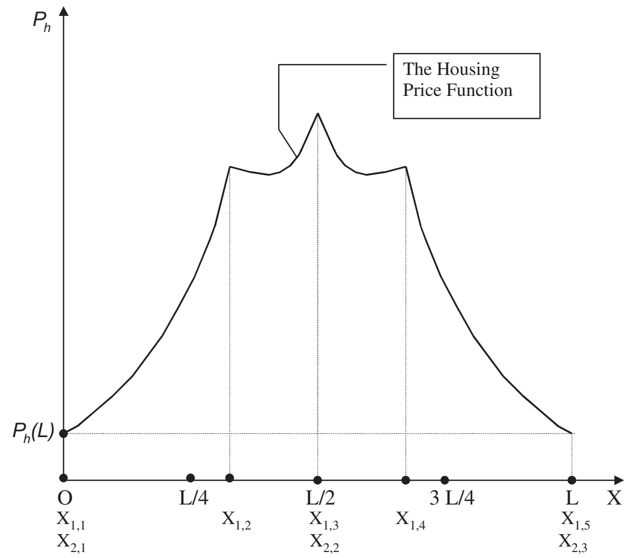

Fig. 4. The housing price function in a complex with the configuration (2,1).

Proof. Suppose that i and i' are two clubs with proportional transportation costs, i.e., $t_i(x)/t_{i'}(x) = \alpha_{ii'}, \forall x$, where $\alpha_{ii'}$, the factor of proportionality, is constant. Then the proportionality is retained by the derivatives as well as by the functions and $t_i'(x) = \alpha_{ii'} t_{i'}'(x)$. Thus, if (20) holds for club i, it holds for its proportional club i' at the same facility location as well. To see this consider (20) for club i, in which we substitute $\alpha_{ii'} t_{i'}'(x)$ for $t_i'(x)$ and then eliminate the proportionality factor $\alpha_{ii'}$ from the equation to obtain (20) for club i' at the same facility location as club i.■

Note that all linear transportation cost functions are proportional and therefore DF clubs with linear transportation cost functions agglomerate perfectly at a single location. If transportation cost functions of various DF clubs are non-linear and non-proportional, then different DF clubs have facilities grouped in a cluster but in separated locations.

Fig. 4 depicts the housing price function in an optimal complex with the configuration (1,2).

We can now summarize the analysis performed in this section in the following Proposition;

Proposition 9. *In an optimal allocation with the basic configuration specified in* (27)

(i) *Facilities of SF clubs in the complex, i.e., of clubs $i \in (1,\ldots,I_1)$, are all perfectly agglomerated in the center of the complex.*
(ii) *Facilities of DF clubs, i.e., of clubs $i \in (I_1 + 1,\ldots,I)$, agglomerate imperfectly in clusters that gravitate towards the center of the complex, i.e., the clusters agglomerate in the second and third quarters of the complex.*
(*iii*) *Average density of population between the clusters of the DF clubs is higher than the average density between these clusters and the boundaries of the complex.*
(*iv*) *If, in a cluster, two DF clubs have proportional transportation cost functions, they share the same facility location.*

It should be noted that although in Fig. 4 the housing price function peaks in the middle of the complex where facilities of clubs with odd number of facilities agglomerate perfectly, in general this may not be the case. If commuting costs to the clubs in the complex center are relatively low and so is their number, then there may not be a peak in the housing price function at the center. It follows that the lower is the housing price function at the center, the more concentrated and closer to the middle of their market areas the clusters become.

5.4. Characterizing the general configuration

In this section we extend some of the results of the previous sections to a model with a general configuration. We first divide the I clubs of the complex into S groups. Each group contains clubs that have the same number of facilities in the complex. Without loss of generality we assume that the indexes of clubs in the same group are adjacent to each other in the complex configuration, as it is in (28).

$$\left(\overbrace{m_1,\ldots,m_1}^{k_1},\ldots,\overbrace{m_S,\ldots,m_S}^{k_S}\right), \quad \text{where } \sum_{s=1}^{S} k_s = I$$
$$\text{and } m_s \neq m_k \text{ if } s \neq k, \quad \forall s,\ k \in (1,\ldots,S),\ \ 1 \leqslant S \leqslant I. \tag{28}$$

In (28), the first group of clubs has k_1 clubs, each having m_1 facilities, the second group has k_2 clubs each with m_2 facilities and so on up to group S in which there are k_S clubs, each having m_S facilities. Note that the indexing of m, the number of facilities of a club in a complex is changed in (28) and now obtains the group index instead of the individual club's index. In the previous notation of the complex configuration the index i of m_i is the club's place in the configuration while in (28) it is the group's place. Thus, while the notation in the configuration $\overbrace{(1,\ldots,1)}^{I}$ is $m_1 = \cdots = m_I = 1$, in the notation of (28) it is $S = 1$, $k_1(\equiv k_S) = I$ and $m_1(=m_S) = 1$. The configuration presented by (27) in the notation of (28) is described by $S = 2$, $k_1 = I_1$ and $k_2(\equiv k_S) = I - I_1$, $m_1 = 1$ and $m_2(\equiv m_S) = 2$. Note that any basic configuration can be presented by the notation of (28). For example, consider the configuration (1,2,3,4,5), which is a basic configuration since the GCD of its entries is one. In this configuration there are five clubs in a complex (i.e., $I = 5$): in club 1 there is one facility per complex (i.e., $m_1 = 1$), in club 2 there are two facilities per complex (i.e., $m_2 = 2$) and so on up to club 5, which has five facilities per complex (i.e., $m_5 = 5$). In (28), $S = I = 5$, $k_1 = \cdots = k_5(\equiv k_S) = 1$, $m_1 = 1, m_2 = 2, \ldots, m_5(\equiv m_S) = 5$. The general configuration (28) may be described as consisting of S pairs of indexes (k_s, m_s).

We can now formulate the Proposition below,

Proposition 10. *Consider a complex with the general configuration* (28)*. Then (1) for each group of clubs s the area of the complex is divided into m_s market areas, each of which is shared by k_s facilities of the group of clubs s, $s = 1,\ldots,S$. (2) There is always at least one group of clubs for which m_s is an odd number. Each group s_o with an odd m_{s_o} must have a middle-facility (i.e., facility $\frac{m_{s_o}+1}{2}$) that is located in the middle of the complex (i.e., at $x_{s_o,m_{s_o}+1}$ shared by all k_{s_o} clubs in group s_o, note that s_o is now the index of $x_{s_o,m_{s_o}+1}$ rather than the index of the individual club). All the k_s middle-facilities of a group with m_s an odd number agglomerate perfectly in the center of the complex and share a joint market area that is symmetric around the middle of the complex. It should be noted that all groups of clubs with an odd number of facilities share the middle of the complex as the location of their respective middle-facilities, however, each group has a middle market area with different boundaries. (3) All the facilities of a group with m_s an odd number, which are not middle-facilities (i.e., all facilities $j, j \neq \frac{m_s+1}{2}$) agglomerate imperfectly in clusters in their respective joint market areas. (4) Clubs of a group with m_s an even number have no middle-facility and at the center of the complex is located the boundary $j = m_s + 1$ at $x_{s,,m_s+1}$, which separates the market areas of facilities $\frac{m_s}{2}$ and $\frac{m_s}{2} + 1$ of all k_s clubs of group s. Facilities of a group of an even m_s agglomerate imperfectly in clusters in their joint market areas. (5) Clubs of group s that have commuting costs proportional to each other agglomerate perfectly at one point in each* *of their joint market areas. (6) Each cluster of agglomerated facilities (perfectly or imperfectly) in segments where the rent function is monotonously increasing towards the center of the complex, gravitates away from the center of its market area and towards the center of the complex. (7) Let* $\underline{s}$ *be the index of the group with the lowest number of facilities that is larger than one, (i.e.,* $\underline{m} = \min(m_s;\ \forall m_s \geqslant 2)$*) and* $\underline{m} = m_{\underline{s}}$.[33] *Then the average density of population between the two clusters of group* $\underline{s}$ *at the edges of the complex (the closest to the complex's boundaries) is higher than the average density between each of these two clusters and each with its closest boundary of the complex.*

The above Proposition includes extensions of assertions pertaining to the configurations $(\overbrace{1,\ldots,1}^{I})$ and (27). The Proposition is proved by showing that the locational patterns specified fulfill the necessary conditions. The proofs follow the same lines as those in the previous sections. For further details on the proof, see Appendix 8.5.1 in the Web Appendixes.

An interesting Corollary follows below:

Corollary 11. *If in the configuration (28) the quotient* $m_s/m_{s'} > 1$ *is an integer, then the market areas of group* s' *clubs are each divided into* $m_s/m_{s'}$ *market areas of group s clubs so that each of the* $m_{s'}$ *sets of* $m_s/m_{s'}$ *market areas of group s are fully imbedded in one market area of group* s'*, where by fully imbedded we mean that* $m_s/m_{s'}$ *market areas of group s fit exactly into a single market area of group* s'*, fully covering it without any spillover.*

To clarify the Corollary consider the following example in which the club configuration is $\left(\overbrace{1,\ldots,1}^{k_1},\overbrace{2,\ldots,2}^{k_2},\overbrace{6,\ldots,6}^{k_3}\right)$. The complex as a whole is the market area of the k_1 facilities of group 1 clubs that have one facility per club (*i.e.*, $m_1 = 1$). The complex is then divided into two symmetric market areas each with k_2 facilities of group 2 clubs (i.e., $m_2 = 2$). Then each market area of group two is divided into three market areas, each containing k_3 facilities of of group 3 clubs (i.e., $m_3 = 6$). Note that the only requirement from the number of clubs in a group, k_s is that it is a positive integer.

We now turn to discuss briefly how to attain global optimum solutions.

5.5. Global optimum

So far we concentrated on optimal allocations with predetermined configurations which we referred to as local optima. One exception, however, is the example in Section 5.2, where we constructed a domain of functions for which the configuration $(\overbrace{1,\ldots,1}^{I})$ is part of the global optimum. In order to obtain additional domains of functions with which other configurations are part of a global optimum, we start from the domain in which the configuration $(\overbrace{1,\ldots,1}^{I})$ is part of the global optimum and change a single parameter in a single function until we obtain a function for which a different configuration is in the global optimum.

To demonstrate, we begin with the commuting cost function for a facility of club 2, i.e., $t_2(y)$, where y is the distance between the household's home and the facility to which it travels. We multiply $t_2(y)$ by α to obtain $\alpha t_2(y)$ as the commuting cost function of club 2. For $\alpha = 1$, the commuting costs are unchanged. We now increase α gradually and with it the commuting costs until eventually a threshold $\alpha_1, 1 < \alpha_1 < \infty$ is reached, at which point it pays to construct an additional facility of type 2 in order to shorten commuting distances. However, adding just one more facility may not be the global optimum since the population distribution changes as well when we add a facility. Therefore, it is possible that the optimum optimorum involves more than just one additional club 2 facility. Actually, when the number of club 2 facilities changes, the global optimum may also involve one more (or less) facility of other clubs as well in which nothing is changed. By continuing to increase α beyond α_1, we go through a series of α-thresholds, each of which with its global optimum that involves additional facilities of club 2 in the complex (possibly changes in the number of other clubs as well). For sufficiently large α, a threshold is reached in which commuting costs for club 2 are so high that it is worthwhile to provide club 2's CG to households at their homes as a private good. At first, for relatively low α, possibly only to households at sparsly populated areas, and at densely populated areas club facilities still exist. A further increase of α to its final threshold causes all the population to get their CG of club 2 at home as a private good.

The above process can be repeated for all transportation cost functions. A similar process can be used for the provision cost functions and for each of the CGs in the utility function. By performing these processes we obtain a variety of configurations in global optima. Note, however, that not all possible configurations are necessarily global optima solutions. At this stage, it seems that to actually obtain global optimum solutions and characterize them, simulation models have to be used.

6. Summary and concluding remarks

The purpose of this paper was to characterize optimal allocations of an economy with spatial clubs and to investigate the concentration of households around facilities and the agglomerations of club facilities in centers. We also investigated ways to decentralize the optimal allocation. Our results showed that each local optimum could be decentralized, sometimes in more than one way. One method of decentralization involved both taxation and regulation and could be applied to most clubs and it seemed to have similarities with real-life practises. Our main findings were that a primary agglomeration of club goods into facilities occurred due to scale economies in the provision of CGs and that it led to a secondary concentration of population, which, in turn, led to a tertiary agglomeration of facilities of different clubs in centers. The three types of agglomerations occurred simultaneously and their ordering is due to causality not timing. Furthermore, we showed that an optimal allocation would never have a uniform population distribution and neither would an allocation with a uniform distribution of population have an effective agglomeration of facilities. We also compared between the effects that travel time and congestion had on the number and sizes of facilities as well as on complex size.

We characterized in detail two types of facility agglomerations, each with a specific complex configuration, one termed *perfect agglomeration* and the other *imperfect agglomeration*. In the perfect agglomeration, facilities of different clubs agglomerated perfectly in the center of the complex, where they were adjacent to each other without residential activity between them. In the imperfect agglomeration facilities of different types of clubs agglomerated imperfectly in clusters symmetrically located around the center of the complex but away from it. The clusters may had households residing between facilities if transportation costs to different clubs were not proportional. Although these clusters were located away from the center of the complex, each of them was drawn away from the center of its market area and towards the center of the complex. We then argued that these two kinds of facility agglomerations also typify agglomerations in general.

[33] i.e., $\underline{m}$ is not defined for the configuration $(1,\ldots,1)$. For $(1,\ldots,1,2,\ldots,2)$, $\underline{m} = 2$ and for (1,3,6), $\underline{m} = 3$ and $\underline{s} = 2$.

The global optimum we investigated was the solution to the model with functions specified in the example in Section 5.2 and the configuration $(1,\ldots,1)$ was part of this global solution. We showed a way to obtain function domains to other global optima with complex configurations different from $(1,\ldots,1)$, by making changes in one of the cost functions (transportation or provision).

One avenue for future research could focus on the relation between certain costs and utility function, and their global optimal configuration. Such a research may shed light on questions like what functions would result in a hierarchy of clubs or what causes facilities of certain types of clubs to be imbedded in facilities of other club types. At these stage it seems that such questions can be resolved only by running computer simulations.

Appendix A. Characterizing the bid housing price and other related functions (for publication)

The following differentiation of the bid housing price function proves Lemma 2. Differentiating (22) with respect to distance, bearing in mind that no facility is located in x, yields the Muthian spatial equilibrium condition,[34]

$$h(x,p_h^b)\dot{p}_h^b(x,j^1,\ldots,j^I)+\dot{T}r(x,j^1,\ldots,j^I)=0$$
$$\text{where } \dot{T}r(x,j^1,\ldots,j^I)=\sum_{i=1}^{I}t_i'(|x-x_{i,2j^i}|)sign(x-x_{i,2j^i}). \tag{B1}$$

A dot above a function designates differentiation with respect to x. The reader should bear in mind that according to our assumptions, $t_i'(y)=\frac{dt_i(y)}{dy}>0$. and $t_i''(y)=\frac{d^2t_i(y)}{dy^2}\leqslant 0$.

Eq. (B1) implies that a marginal displacement at a given location causes a marginal change in the bid housing price function proportional to the sum of all marginal changes in the home-facility commuting costs to the facilities of clubs $j_1,\ldots,j_I$. The factor of proportionality is $-1/h(x|p_h^b(x))$, i.e., minus the reciprocal of the amount of housing consumed by a household at x, provided $p_h^b(x)$ is the price of housing. Note that since $t_i(|y|)$ is not differentiable at $y=0$, at the facility locations, $x_{i,2j_i}, p_h^b(x/(j^i))$ is continuous but not differentiable. For an x that is not a facility location, the second derivative of the bid housing price is obtained by differentiating (B1) with respect to distance. Thus

$$\ddot{p}_h^b=-\frac{\frac{\partial h}{\partial p_h^b}\left(\dot{p}_h^b\right)^2+\sum_{i=1}^{I}t_i''(|x-x_{i,2j_i}|)}{h(\cdot)}\geqslant 0. \tag{B2}$$

Consequently, (B2) implies that $p_h^b(x)$ is a concave function of x.

Since the housing price function, $p_h(x)$, at a location x that is not a node coincides with one of the bid rent functions, it has all the properties of a bid housing price function, except at boundaries and facility locations where it is continuous but not differentiable. We now turn to other continuous functions that depend on $p_h(x)$. By differentiating (14) we obtain

$$\frac{dH^s}{dp_h}=\frac{1}{c_h''(H^s)}>0\Rightarrow \dot{H}^s=\frac{dH^s}{dp_h}\dot{p}_h=sign(\dot{p}_h)\frac{dH^s}{dp_h}|\dot{p}_h| \tag{B3}$$

Eq. (B3) implies that the supply of housing at a given location is an increasing function of its product's price there, and that $\dot{H}^s$ has the same sign as $\dot{p}_h$.

The density function, $n(p_h)=H^s(p_h(x))/h(p_h(x))$ (defined as the number of households per unit of land) increases with the price of housing. To see this, we make the following differentiation:

[34] The function $sign(x)$ is given by $sign(x)=\begin{cases}1,\text{ for } x>0\\ 0 \text{ for } x=0\\ -1 \text{ for } x<0\end{cases}$. The function $sign(x)$ is differentiable everywhere except at $x=0$. Furthermore, $|x|=x\cdot sign(x)$ and $\partial|x|/\partial x=sign(x)$, except atr $x=0$, where it is not defined.

$$\frac{\partial n(x)}{dp_h}=\frac{d(H^s/h)}{dp_h}=\frac{h\partial H^s/\partial p_h-H^s\partial h/\partial p_h}{h^2}>0. \tag{B4}$$

The sign of (B4) follows from (B3) and the substitution effect $[\partial h(\cdot)/\partial p_h(\cdot)]_{dU=0}<0$ in (10). From (B4) it follows that the density $n(x)=H^s(x)/h(x)$ increases with distance the same way that $p_h(x)$ does.

By differentiating the land rent function in (15) and using (B1) as well as (14), we obtain that

$$\dot{R}(x)=H^s(x)\dot{p}_h(x). \tag{B5}$$

which implies that $R(x)$ varies with distance in the same way that $p_h(x)$ does. By differentiating $\dot{R}(x)$, we obtain that

$$\ddot{R}(x)=\dot{H}^s\dot{p}_h+H^s\ddot{p}_h\geqslant 0. \tag{B6}$$

Together, Eqs. (B6) and (B2) imply that in the general case R, like p_h, is a concave function of x.

The functions $p_h^b(x)$ and $p_h(x)$ are also functions of the parameters U, Y and G_{ij}. By differentiation of (18) as well as (5), with respect to Y, taking into account that only variables controlled by the consumer may be indirectly affected, namely $H(x)$ and $Z(x)$, we obtain that

$$\frac{\partial p_h(x)}{\partial Y}=\frac{1}{h(x)}\geqslant 0. \tag{B7}$$

In the same way we obtain for G_{ij} that

$$\frac{\partial p_h(x)}{\partial G_{ij}}=\frac{1}{h(x)}\frac{U_{i+2}}{U_1}>0,\quad \forall x,\ x_{i,2j-1}\leqslant x\leqslant x_{i,2j+1}. \tag{B8}$$

Web Appendices. Supplementary data

Supplementary data associated with this article can be found, in the online version, at doi:10.1016/j.jue.2010.07.004.

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